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Review of Matrices

A quick review of matrices with a few examples

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MATRICES

Main concepts of Matrices

<table>
<tr>
<td>What is a matrix?</td>
<td>Another way of showing equations but they are put into a rectangular format. Matrices come in different forms
<table>
<tr><td>Different matrices</td><td>What is its size</td></tr>
<tr><td>$\begin{bmatrix} 1 & 2 & 0 \\ 4 & 3 & 3 \\ 4 & 7 & 8 \end{bmatrix}$</td><td>This is a 3×3 matrix
Why there are 3 rowx and 3 columns</td></tr>
<tr><td>$\begin{bmatrix} 1 & 3 & 5 \end{bmatrix}$</td><td>This is a 1×3 matrix
Why? Since there is 1 row and 3 columns</td></tr>
<tr><td></td><td></td></tr>
</table>

Remember the rows are first and then the column
Within each matrix we can be very specific about how we identify the elements
$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$
For example notice- so the element in the first row and second column is a_{12}</td>
</tr>
<tr>
<td>Where are matrices used?</td>
<td>They are used in a large variety of field ranging from mathematics, physics, and even economic modeling
They are also used in Probability which is where we will see them being used as transition matrices

In methods we will use them to solve equations and transformation equations under a number of different transformations.</td>
</tr>
<tr>
<td>Concept-1</td>
<td>Two matrices are equal if the number :
• Each has the same number of rows and the same number of columns
• Each of the elements are equal to each other
$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

So these two matrices are equal to each other as they have the same rows and columns , therefore the values are: a = 1, b = 2, c = 3 and d = 4</td>
</tr>
<tr>
<td>Concept-2</td>
<td>The size or dimension of a matrix is specified by the number of rows (m) and the columns of (n). The dimension is written as $m\times n$

Examples
<table>
<tr><td>Matrix</td><td>Dimension or size of a matrix
[the number of rows (m) and the columns of (n)]</td></tr>
</table></td>
</tr>
</table>

<table>
<tr>
<td></td>
<td>
<table>
<tr><td>$\begin{bmatrix} 1 & 3 \\ -2 & 5 \\ 3 & 6 \end{bmatrix}$</td><td>$m \times n = 3 \times 2$</td></tr>
<tr><td>$\begin{bmatrix} 1 & 2 & 5 & -2 & 7 \end{bmatrix}$</td><td>$m \times n = 3 \times 2$</td></tr>
<tr><td>$\begin{bmatrix} 1 & 2 & 4 & 7 \\ 2 & 4 & 9 & 0 \\ -2 & 3 & 5 & 6 \end{bmatrix}$</td><td>$m \times n = 4 \times 4$</td></tr>
</table>
So we need to be careful that we do not confuse the way matrices are expressed – remember rows are first followed by columns!
</td>
</tr>
<tr><td></td><td></td></tr>
<tr>
<td>Concept- 3</td>
<td>
Concept-3-Addition or subtraction of matrices

We can add or subtract similar matrices provided they have the same dimension, in other words are the same

Example
<table>
<tr><td>$\begin{bmatrix} 1 & 3 \\ -2 & 5 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 1 & 25 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 1+2 & 1+3 \\ -2+1 & 5+25 \\ 3+4 & 6+7 \end{bmatrix}$</td><td>Notice how we added each individual element</td></tr>
</table>
We must be careful when adding or subtracting- if not sure use the graphics calculator
</td>
</tr>
<tr>
<td>Concept- 4</td>
<td>
Multiplication of a matrix by a number
Normally we can multiply a matrix by a real number and all we will need to do is multiply the real number by each of the elements of the matrix
$3 \times \begin{bmatrix} 1 & 3 \\ -2 & 5 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 3\times1 & 3\times3 \\ 3\times-2 & 3\times5 \\ 3\times3 & 3\times6 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ -6 & 15 \\ 9 & 18 \end{bmatrix}$

Notice that I multiplied every element on the left side of the equation.
</td>
</tr>
<tr>
<td>Concepts -5</td>
<td>
Solving simple equations using matrices

Take the example below

Question-Find $2A + X = B$ given the following matrices
$A = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -4 \\ -2 & 8 \end{bmatrix}$
Solution
<table>
<tr><th>Steps</th><th>Method</th></tr>
<tr><td>Rewrite the equation</td><td>If we rewrite the equation we get the following
$2A + X = B$
$X = B - 2A$</td></tr>
</table>
</td>
</tr>
</table>

Now we put the matrices down	$X = \begin{bmatrix} 0 & -4 \\ -2 & 8 \end{bmatrix} - 2\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$
Now expand the second part	$X = \begin{bmatrix} 0 & -4 \\ -2 & 8 \end{bmatrix} - \begin{bmatrix} 6 & 4 \\ -2 & 2 \end{bmatrix}$
Now solve it	$X = \begin{bmatrix} -6 & -8 \\ 0 & 6 \end{bmatrix}$

So we need to go through the steps one at a time. If we use a graphics calculator then there is no problem

Concept- 6 Multiplying matrices together

This is where it gets a little tricky especially when you do it by hand and do not use graphics calculator

Firstly a few rules
We can only multiply two matrices if they meet the following condition- the columns of the first matrix is equal to the rows of the second matrix!

Example

Question- Find AB given that $A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

Solution

Firstly ask yourself if this matrix is defined?
Remember the condition- the column of the first matrix should be equal to rows of second matrix

	rows	column
matrix A	2	2
matrix B	2	1

So now we can proceed with the multiplication

It is easier to write it as follows - $A \times B = (2 \times 2) \times (2 \times 1)$, notice how easy it is to see if the multiplication is defined

Now how do we multiply these two matrices
Answer- the rows of the first matrix with the colum of the first row and so on … watch

$$AB = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \times \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} (2 \times 5) + (4 \times 3) \\ (3 \times 5) + (6 \times 3) \end{bmatrix} = \begin{bmatrix} 22 \\ 33 \end{bmatrix}$$

Notice we obtain a 2 row by 1 column matrix.
We could had seen that from the test we did to see if the multiplication will work out

$$A \times B = (2 \times 2) \times (2 \times 1)$$

So the resulting matrix would be 2×1 matrix

Concept-7 identities

A matrix with the same number of rows and columns is called a square matrix. For square matrices there exists the multiplicative identity ($\boldsymbol{I}$)

<table>
<tr>
<td></td>
<td>The multiplicative identity for a 2×2 is the following $I=\begin{bmatrix}1&0\\0&1\end{bmatrix}$
What is special about an identity matrix?
$AI=IA=A$ is true for any square matrix
which leads us to working out the inverses and the determinants of matrices</td>
</tr>
<tr>
<td>Concept-8-
inverses and
determinants</td>
<td>The inverse of a matrix A is a matrix called B if it produces the following result
$AB=BA=I$
This matrix, B is normally written as follows A^{-1}

So this is what we obtain then
If we have a matrix A then the inverse can be found by doing the following

$$A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$$
$$A^{-1}=\begin{bmatrix}\frac{d}{ad-bc}&\frac{-b}{ad-bc}\\\frac{-c}{ad-bc}&\frac{a}{ad-bc}\end{bmatrix}=\frac{1}{\det(A)}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$$
The determinant is found as $\det(A)=ad-bc$

A square matrix is said to be regular if its inverse exists. The square matrices that do not have an inverse are called singular matrices.</td>
</tr>
<tr>
<td>Problem involving
inverses</td>
<td>Example- Find the inverse of the following matrix $A=\begin{bmatrix}3&-2\\7&4\end{bmatrix}$

<table>
<tr>
<td>Step -1-find the determinant by using the formula $\det(A)=ad-bc$</td>
<td>$\det(A)=ad-bc$
$\det(A)=(3\times 4)-(-2\times -7)$
$\det(A)=12--14=26$</td>
</tr>
<tr>
<td>Now you can write the new inverse matrix
Noticing that you will need to
$A^{-1}=\frac{1}{\det(A)}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$</td>
<td>$A^{-1}=\frac{1}{\det(A)}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$
$A^{-1}=\frac{1}{26}\begin{bmatrix}4&2\\-7&3\end{bmatrix}$</td>
</tr>
</table>

Be very careful with signs especially when doing these calculations manually

Why go to so much trouble then? Because using this idea we can solve equations and very complicated equations that normally would take some time to do using the algebraic methods learnt in the past

Problem

Solve the following simultaneous equations</td>
</tr>
</table>

$$3x-2y=6$$
$$7x+4y=7$$

Solution

We can express these simultaneous equations in matrix form (prove to yourselfs that this is correct)

$$\begin{bmatrix} 3 & -2 \\ 7 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 6 \\ 7 \end{bmatrix}$$

This is of the form $AX=B$ which we can find the inverse by doing the following

$$A^{-1}AX=A^{-1}B$$
$$X=A^{-1}B$$

Notice you will need to find the inverse of A and multiply it by B

We already know the inverse of A which is $A^{-1}=\frac{1}{26}\begin{bmatrix} 4 & 2 \\ -7 & 3 \end{bmatrix}$

So the answer is therefore the following:

$$X=A^{-1}B$$

$$X=\frac{1}{26}\begin{bmatrix} 4 & 2 \\ -7 & 3 \end{bmatrix}\begin{bmatrix} 6 \\ 7 \end{bmatrix}$$
$$X=\frac{1}{26}\begin{bmatrix} 4 & 2 \\ -7 & 3 \end{bmatrix}\begin{bmatrix} (4\times 6)+(2\times 7) \\ (-7\times 6)+(3\times 7) \end{bmatrix}=\frac{1}{26}\begin{bmatrix} 38 \\ -21 \end{bmatrix}$$

Concept- singular matrices and solutions

Remember that a 2×2 matrix is said to be singular if its determinant is equal to 0
The matrix being singular can correspond to one of two situations:
Situation-1- there are infinitely many solutions
Situation-2- there is no solution

Let us examine this situation now

Problem-1 –Explain why the simultaneous equations $2x+3y=6$ and $4x+6y=24$ have no solutions.

Solution

If you think about it we can understand why two straight lines will have no solution. This can only happen if the lines do not intersect at all. For that to happen they would have the same gradient

Let us see if the gradient is the same

Steps to take	Equation 1	Equation 2
	$2x+3y=6$	$4x+6y=24$
Rearrange equations with y being the subject	$2x+3y=6$ $3y=-2x+6$ $y=\frac{-2}{3}x+\frac{6}{3}$	$4x+6y=24$ $6y=-4x+24$ $y=\frac{-4}{6}x+\frac{24}{6}$ $y=\frac{-2}{3}x+\frac{24}{6}$

So notice how these two lines have the same gradient so they will never intersect each other, so no solutions will exist.
But we could had arrived at that conclusion by trying to solve these two equations using the matrix method

Steps to take	Express into matrix format
$2x+3y=6$ $4x+6y=24$	$\begin{bmatrix}2 & 3\\4 & 6\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}6\\24\end{bmatrix}$
Now we need to find the determinant of the front matrix $A=\begin{bmatrix}a & b\\c & d\end{bmatrix}$ $\det(A)=ad-bc$	$\det(A)=ad-bc$ $\det(A)=(2\times 6)-(3\times 4)$ $\det(A)=12-12=0$ Since determinant is zero we can have two possible situations- infinitely many solutions or no solution

So in this case there are no solutions.
If the two equations were the same then we would have the second case which would mean infinitely many solutions!

Problem using simultaneous equations

Consider the following simultaneous equations

$$(m-2)x+y=2$$
$$mx+2y=k$$

Find the values of m and k such that the system of equations has:

a) A unique solution
b) No solution
c) Infinitely many solutions

We can obviously use our graphics calculator but let us do it manually

First step let us express the equations in the normal format i.e with y the subject

This is what we have to start with	$(m-2)x+y=2$	$mx+2y=k$
Put them into y format	$(m-2)x+y=2$ $y=-(m-2)x+2$	$mx+2y=k$ $2y=-mx+k$ $y=\frac{-m}{2}x+\frac{k}{2}$

No lets us look at the first situation- we want a unique solution
What this means is that the two lines intersect each other only once!

Part-1

For this to happen the gradient of both lines must not be equal to each other , otherwise there will infinitely many solutions

$$-(m-2)=\frac{-m}{2}$$

$$-m+2=\frac{-m}{2}$$

so $-2m+4=-m$

$$-m+4=0$$

$$m=4$$

So m cannot be equal to 4, otherwise we would have infinitely many solutions
So for a unique solution you can have any value of m except 4 and k any real number

Part-2

No solution
This is where the gradients of both lines are the same but the value of k is different for both of the equations
therefore

$$2=\frac{k}{2}$$

$$k=4$$

So k must not be equal to 4 and we would obtain a unique solutions

Part-3

Infinitely many solutions

For this to occur both equations must be identical equal to each other- the same gradient and the same c value

So if m = 4 and k = 4 then we would have the same equations giving us infinitely many solutions

So if we can just think back to what it means to solve simultaneous equations we can then find the relevant solution.

If we need to solve 3 simultaneous equations we will then use a graphic calculator or use the matrix method and solve these equations

Example:
Solve the following equations

$$2x+y+z=-1$$

$$3y+4z=-7$$

$$6x+z=8$$

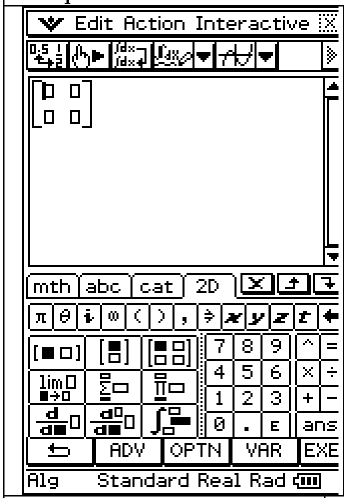
Solution using matrix methods

Equation given	$2x+y+z=-1$ $3y+4z=-7$ $6x+z=8$
Set the equation up properly- always have x first then y and then z NOTICE I ADDED	$2x+y+z=-1$ $0x+3y+4z=-7$ $6x+0y+z=8$
Now we can set up the equation $AX=B$	$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 4 \\ 6 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -1 \\ -7 \\ 8 \end{bmatrix}$ $AX=B$ $A^{-1}AX=A^{-1}B$ $X=A^{-1}B$
Now we use the graphics calculator to find the solution as I have not shown you how to find the determinant of a three by three matrix.	$\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix}$

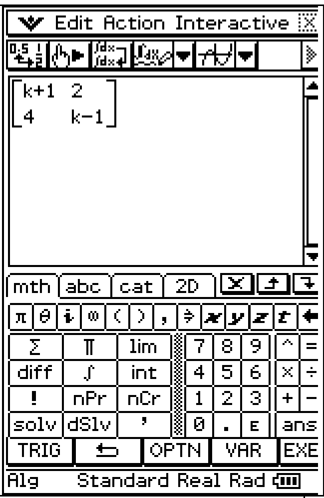
Equations and Matrices

Determining when a system of equations has a unique solution

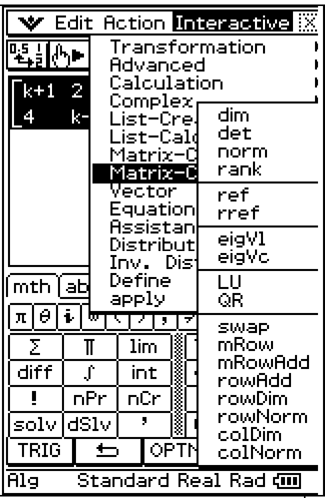
Consider the linear equations given by the following: $\begin{bmatrix} k+1 & 2 \\ 4 & k-1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ k \end{bmatrix}$, find value of k for a unique solution



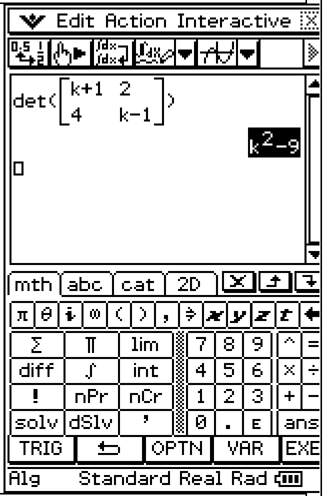
Go to Main area and use 2D and use Matrix template



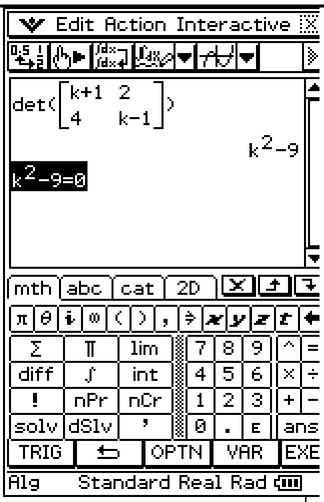
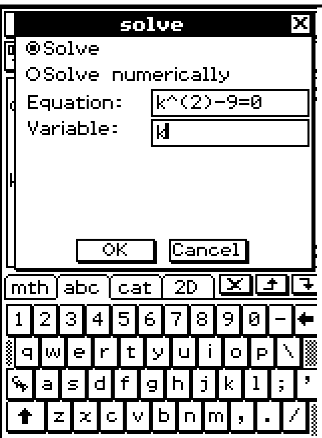
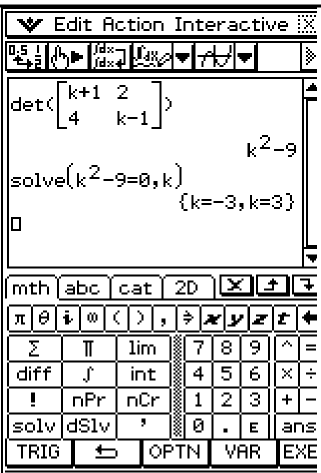
Now enter the information carefully. Remember to use abc keys



Now select the matrix and
Go interactive
Matrix-calculation
det



This is what you will obtain

 $\det\left(\begin{bmatrix} k+1 & 2 \\ 4 & k-1 \end{bmatrix}\right)$ k^2-9 $k^2-9=0$	 ● Solve ○ Solve numerically Equation: k^(2)-9=0 Variable: k	 $\det\left(\begin{bmatrix} k+1 & 2 \\ 4 & k-1 \end{bmatrix}\right)$ k^2-9 $\text{solve}(k^2-9=0,k)$ $\{k=-3,k=3\}$	
Select the expression for the determinant and put it onto a new line and set to zero and solve	Highlight it Interactive Equation Solve Remember to put k for variable	And we get the answers. This tells us when the system of equation does not have a unique solution. For all other values of k it will have a unique solution	
