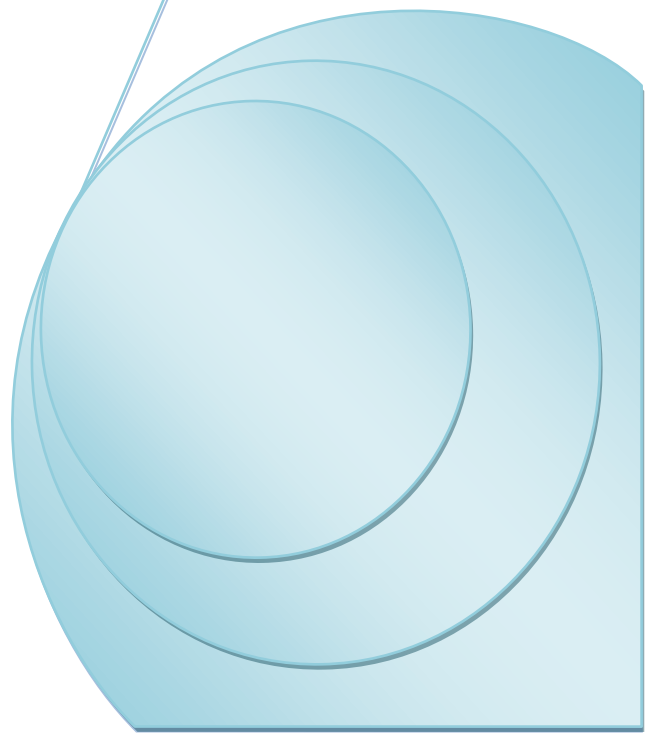
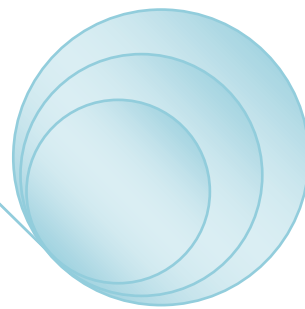
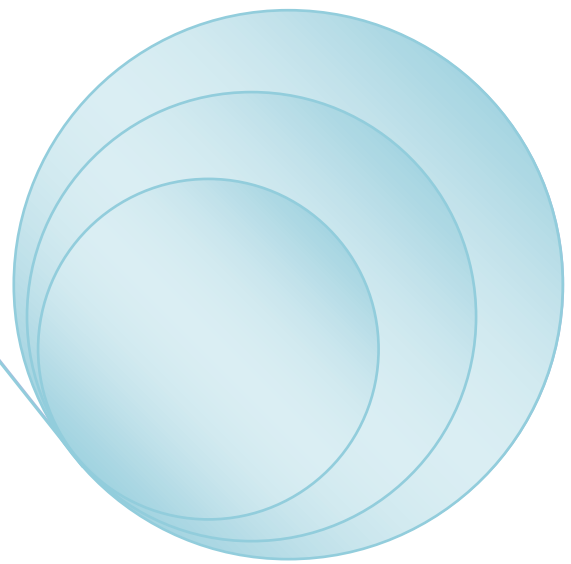


Updated-28 April 2011

All reasonable efforts have been made to make sure the notes are accurate. The author cannot be held responsible for any damages arising from the use of these notes in any fashion

Circular functions

A quick summary of circular functions



Mathematical Methods-Chapter 6-Circular Functions
Introduction to circular functions

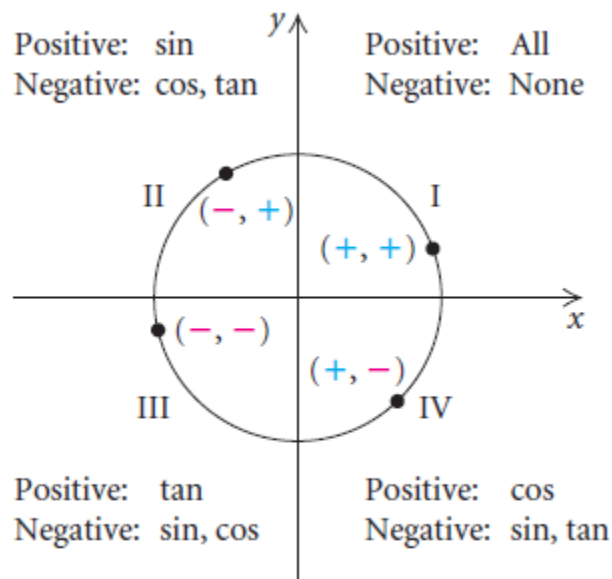
Quick review of the past

Degree and Radian Mode

Radian measure is more commonly used for angles rather than degree measure in this course.

It is extremely important that when you calculate sin, cos or tan of an angle, you always check the mode of your calculator to see if it is in degree or radian mode and adjust it for you calculation.

In general we leave the scientific calculator in degree mode and the graphics calculator in radian mode

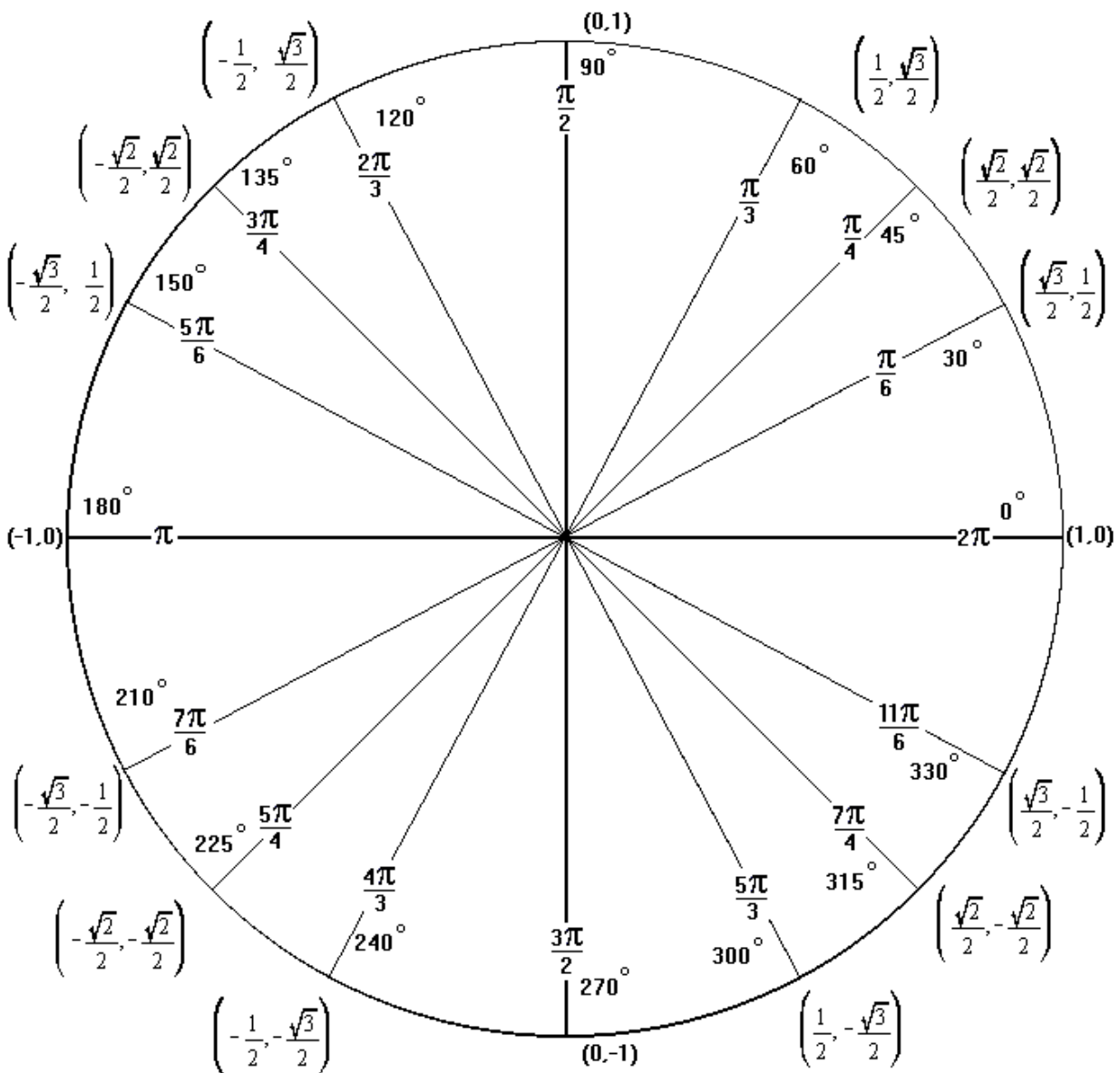


<p>QUADRANT 2 Only sin is positive $\pi - \theta$</p>	<p>QUADRANT 1 All sin, cos, tan are positive θ</p>
<p>QUADRANT 3 Only tan is positive $\pi + \theta$</p>	<p>QUADRANT 4 Only cos is positive $2\pi - \theta$</p>

REMEMBER θ IS MEASURED FROM THE HORIZONTAL AXIS ANTICLOCKWISE!!!

Useful rules connecting degrees and radians

1	π radians is the same angle as 180° - You can think of π as a symbol for a straight line angle
2	To convert degrees to radians, multiply by $\times \frac{\pi}{180}$
3	To convert radians to degrees multiply by $\times \frac{180}{\pi}$
4	It is useful to remember that one radian is just slightly less than 60°



Various Relationships of Trigonometric Functions

$\sin \theta = \cos(\pi - \theta)$	$\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$	$\sin(u + v) = \sin u \cos v + \cos u \sin v$	$\csc \theta = \frac{1}{\sin \theta}$
$\tan \theta = \cot(\pi - \theta)$	$\tan\left(\frac{\pi}{2} - \theta\right) = \cot \theta$	$\sin(u - v) = \sin u \cos v - \cos u \sin v$	$\sec \theta = \frac{1}{\cos \theta}$
$\sec \theta = \csc(\pi - \theta)$	$\sec\left(\frac{\pi}{2} - \theta\right) = \csc \theta$	$\cos(u + v) = \cos u \cos v - \sin u \sin v$	$\cot \theta = \frac{1}{\tan \theta}$
$\cos \theta = \sin(\pi - \theta)$	$\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$	$\cos(u - v) = \cos u \cos v + \sin u \sin v$	$\sin(\theta) = \frac{1}{\csc(\theta)}$
$\cot \theta = \tan(\pi - \theta)$	$\cot\left(\frac{\pi}{2} - \theta\right) = \tan \theta$	$\tan(u + v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$	$\tan \theta = \frac{\sin \theta}{\cos \theta}$
$\csc \theta = \sec(\pi - \theta)$	$\csc\left(\frac{\pi}{2} - \theta\right) = \sec \theta$	$\tan(u - v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$	$\cot \theta = \frac{\cos \theta}{\sin \theta}$
$\sin\left(\theta + \frac{\pi}{2}\right) = \cos \theta$	$\sin(-\theta) = -\sin \theta$	$\sin 2\theta = 2 \sin \theta \cos \theta$	$\tan \theta = \frac{1}{\cot \theta}$
$\sin\left(\theta - \frac{\pi}{2}\right) = -\cos \theta$	$\cos(-\theta) = \cos \theta$	$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$	
$\cos\left(\theta + \frac{\pi}{2}\right) = -\sin \theta$	$\tan(-\theta) = -\tan \theta$	$\cos 2\theta = 1 - 2 \sin^2 \theta$	
$\cos\left(\theta - \frac{\pi}{2}\right) = \sin \theta$	$\sin(\pi + \theta) = -\sin(\theta)$	$\cos 2\theta = 2 \cos^2 \theta - 1$	
$\sin(2\pi - \theta) = -\sin(\theta)$	$\cos(\pi + \theta) = -\cos(\theta)$		
$\cos(2\pi - \theta) = \cos(\theta)$	$\tan(\pi + \theta) = \tan(\theta)$		
$\tan(2\pi - \theta) = -\tan(\theta)$			

$\sin(-\theta) = -\sin \theta$	$\sin\left(\frac{\pi}{2} - \theta\right) = +\cos \theta$	$\sin(\pi - \theta) = +\sin \theta$
$\cos(-\theta) = +\cos \theta$	$\cos\left(\frac{\pi}{2} - \theta\right) = +\sin \theta$	$\cos(\pi - \theta) = -\cos \theta$
$\tan(-\theta) = -\tan \theta$	$\tan\left(\frac{\pi}{2} - \theta\right) = +\cot \theta$	$\tan(\pi - \theta) = -\tan \theta$

Here are some more

$\sin\left(\theta + \frac{\pi}{2}\right) = +\cos \theta$	$\sin(\theta + \pi) = -\sin \theta$	$\sin(\theta + 2\pi) = +\sin \theta$
$\cos\left(\theta + \frac{\pi}{2}\right) = -\sin \theta$	$\cos(\theta + \pi) = -\cos \theta$	$\cos(\theta + 2\pi) = +\cos \theta$
$\tan\left(\theta + \frac{\pi}{2}\right) = -\cot \theta$	$\tan(\theta + \pi) = +\tan \theta$	$\tan(\theta + 2\pi) = +\tan \theta$

Below are some common values that you should know:

θ		$\sin \theta$	$\cos \theta$	$\tan \theta$
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	undefined
180°	π	0	1	0

Example 1

Express $\frac{5\pi}{6}$ in terms of $\frac{\pi}{6}$ and find exact values for $\sin \frac{5\pi}{6}$, $\cos \frac{5\pi}{6}$ and $\tan \frac{5\pi}{6}$

Solution:

Notice that the angle is in quadrant 2

$$\rightarrow \frac{5\pi}{6} = (\pi - \frac{\pi}{6})$$

$$\rightarrow \sin \frac{5\pi}{6} \rightarrow \sin\left(\pi - \frac{\pi}{6}\right) \rightarrow \cos \frac{\pi}{6} \rightarrow \frac{\sqrt{3}}{2}$$

$$\rightarrow \cos \frac{5\pi}{6} \rightarrow \cos\left(\pi - \frac{\pi}{6}\right) \rightarrow -\sin \frac{\pi}{6} \rightarrow -\frac{1}{2}$$

$$\rightarrow \tan \frac{5\pi}{6} \rightarrow \tan\left(\pi - \frac{\pi}{6}\right) \rightarrow -\tan \frac{\pi}{6} \rightarrow -\frac{1}{\sqrt{3}}$$

Example 2

Express $\frac{11\pi}{6}$ in terms of $\frac{\pi}{6}$ and find exact values for $\sin \frac{11\pi}{6}$, $\cos \frac{11\pi}{6}$ and $\tan \frac{11\pi}{6}$

Example 3

Express \sin , \cos , \tan for 155° in terms of a trigonometric function of 25°

Solution:

Notice that the angle is 155° which is in the 2nd Quadrant

$$\therefore \sin 155^\circ = \sin(180^\circ - 25^\circ)$$

$$\rightarrow = \sin 25^\circ$$

$$\rightarrow = 0.4226$$

$$\cos 155^\circ = \cos(180^\circ - 25^\circ)$$

$$\rightarrow = -\cos 25^\circ$$

$$\rightarrow = -0.9063$$

$$\tan 155^\circ = \tan(180^\circ - 25^\circ)$$

$$\rightarrow = -\tan 25^\circ$$

$$\rightarrow = -0.4663$$

Example 4

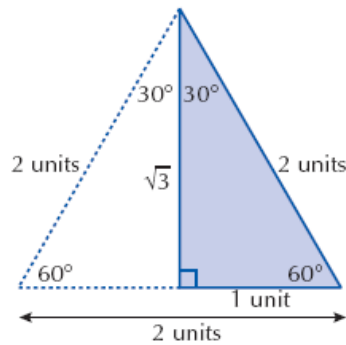
Evaluate \sin , \cos and \tan for angle -25°

Solution:

$$\begin{aligned}\sin(-25^\circ) &= -\sin 25^\circ \\ &= -0.4226\end{aligned}$$

$$\begin{aligned}\cos(-25^\circ) &= \cos 25^\circ \\ &= 0.9063\end{aligned}$$

$$\begin{aligned}\tan(-25^\circ) &= -\tan 25^\circ \\ &= -0.4663\end{aligned}$$



Trigonometric ratio	Value	Trigonometric ratio	Value
$\sin 60^\circ$		$\sin 30^\circ$	
$\cos 60^\circ$		$\cos 60^\circ$	
$\tan 60^\circ$		$\tan 30^\circ$	

This triangle should help you memorize the exact values of certain values of sin, cos and tan

Sketching trigonometric equations and solving them Solving Trigonometric Equations

Many ways of solving trigonometric equations but before we look at a very easy way it would be good to review the sketching of trigonometric functions since one of the ways of checking that the solutions we have obtained are in correct is by looking at the graph.

Definition of Trigonometric Functions

SINE GRAPH

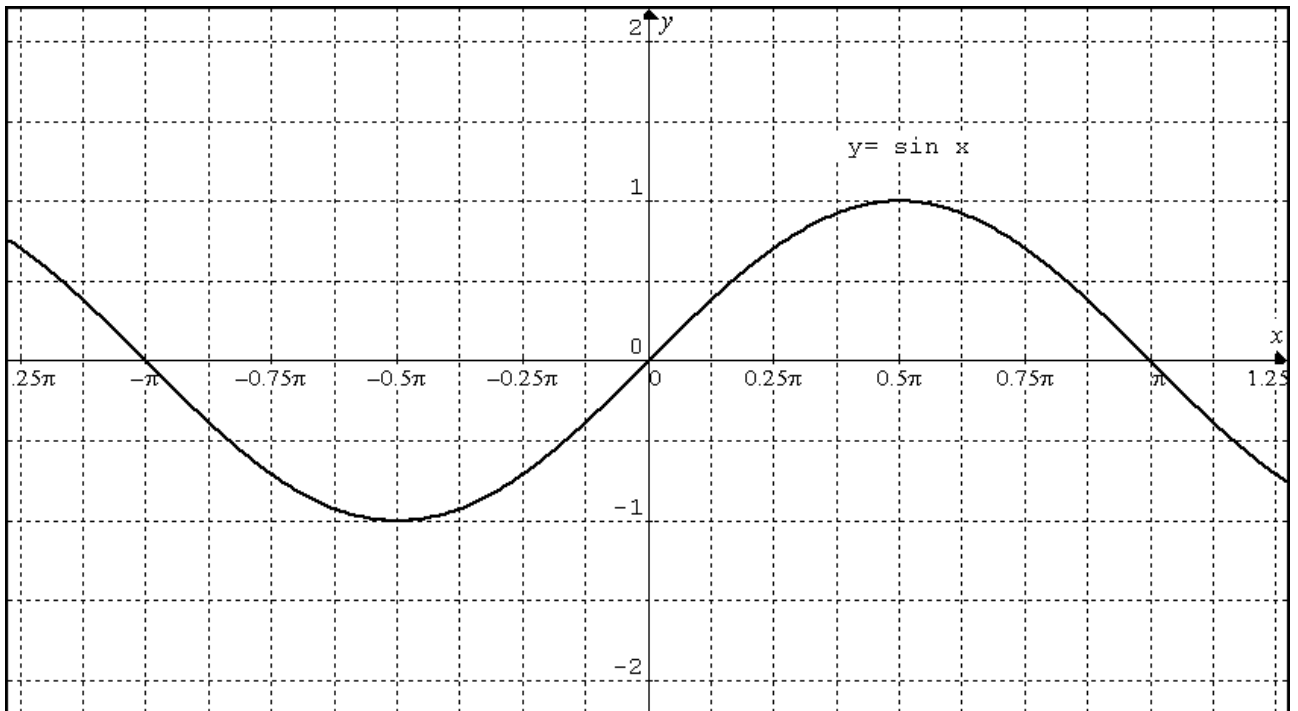
$y = \sin \theta$ Is really defined for the following:

Period: 2π

Amplitude: 1

Domain: $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Let's look at the graph:



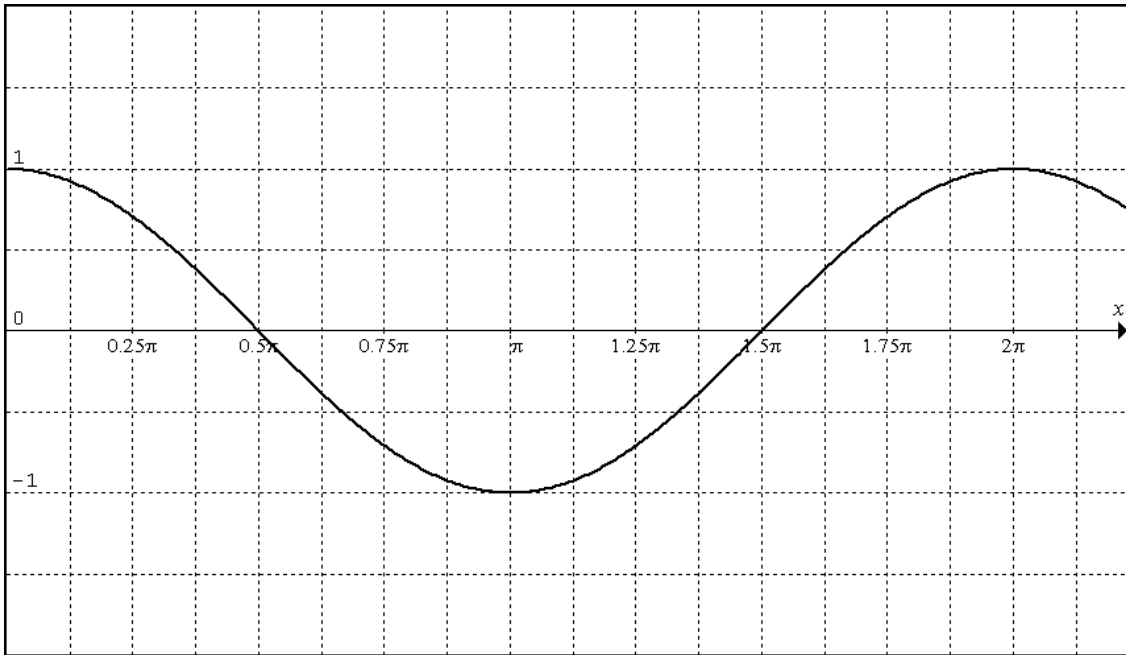
Cosine Graph (cos)

$y = \cos \theta$ Is really defined for the following:

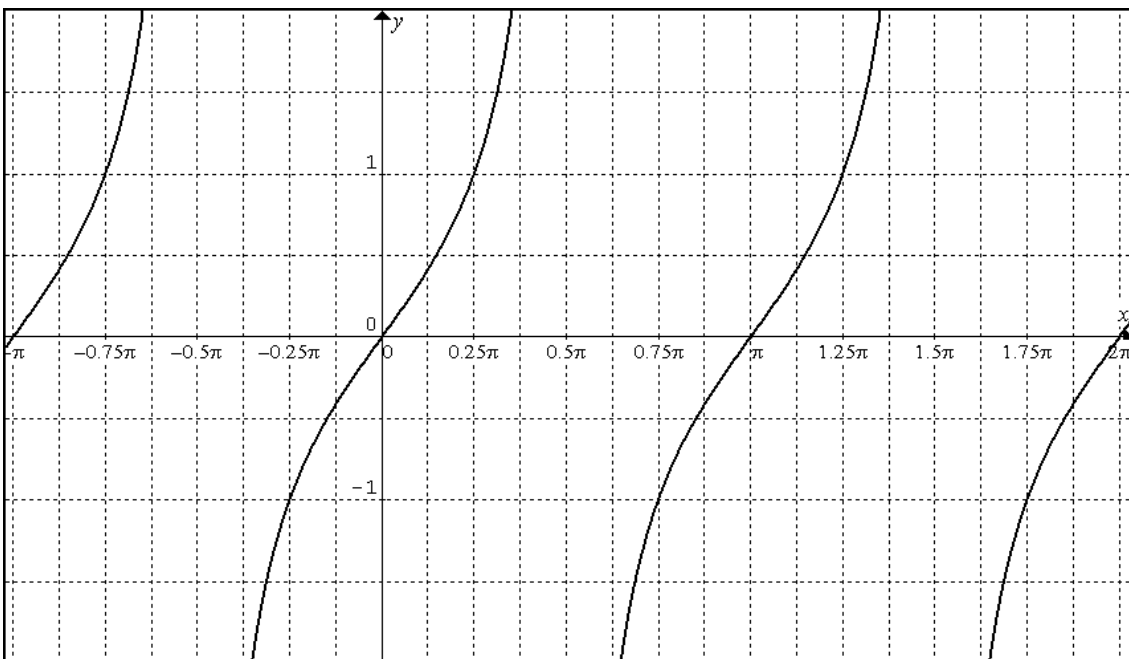
Period: 2π

Amplitude: 1

Domain: $[0, 2\pi]$



$y = \tan \theta$ Is defined for the following: Period π



Let's us have a look at the way transformation affect trigonometric functions

$$y = A \sin a(x+b) + C$$

Now notice what the following numbers do:

A- Amplitude of the graph

a- Period = $\frac{2\pi}{a}$ (this affects the period or the cycles the graph completes)

b- Translation of b to the left (means the graph is moved to the left by b units)

C- Translation of C upwards (means the graph is moved C units upwards)

Lets us look once again at a complete example to understand the significance of the above

So we want to sketch the graph $y = 3 \sin 2(x + \frac{\pi}{4}) + 1$ for the domain $(0, \pi)$

Let's us break it up into various steps:

Step 1: Sketch $y = 3 \sin 2x$

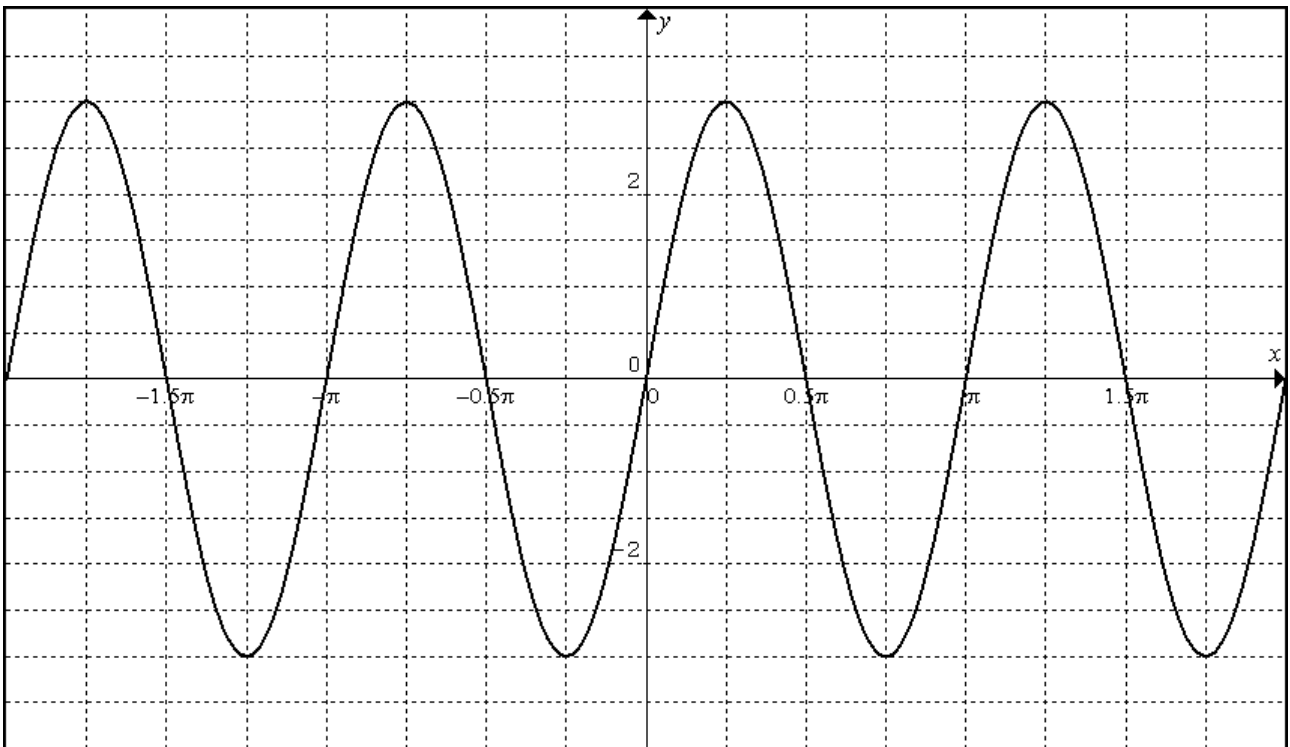
Step 2: Now move it horizontally $\frac{\pi}{4}$ to the left

Step 3: Now move it upwards by $+1$ units and you will have the final graph

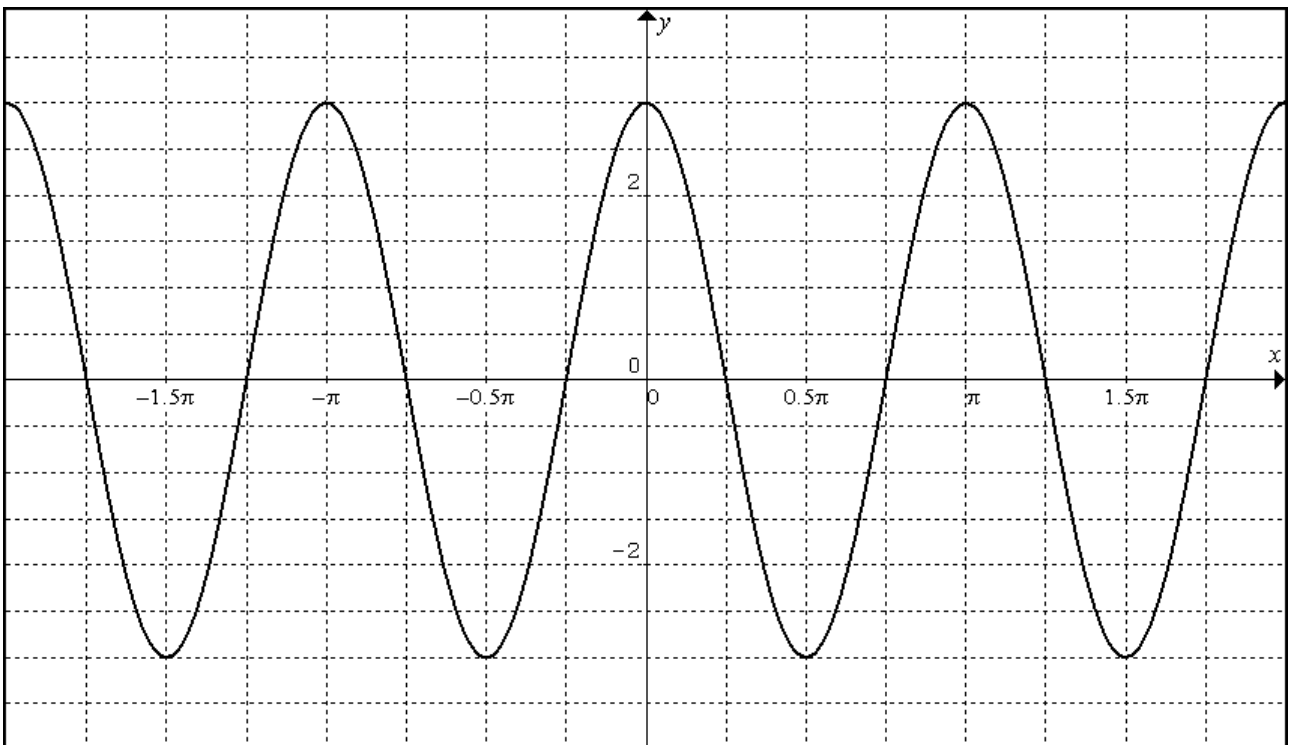
Obviously we can just take out our graphics calculator and put the function in and get the answer immediately but remember we are looking at how the various transformations affect the final graph.

So let look at those transformation (at this moment we are looking at domain $(-2\pi, 2\pi)$
[Yes, I know it is not the domain in question!]

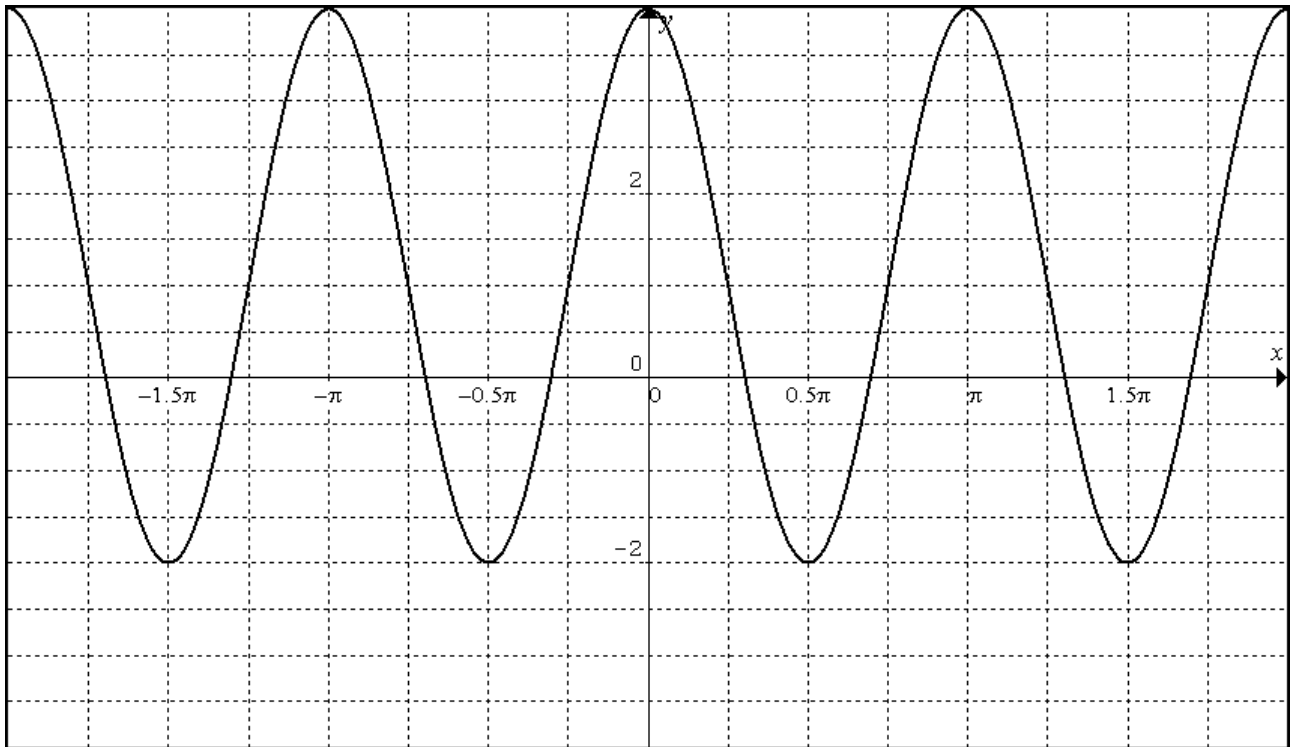
Step 1: Sketch $y = 3 \sin 2x$



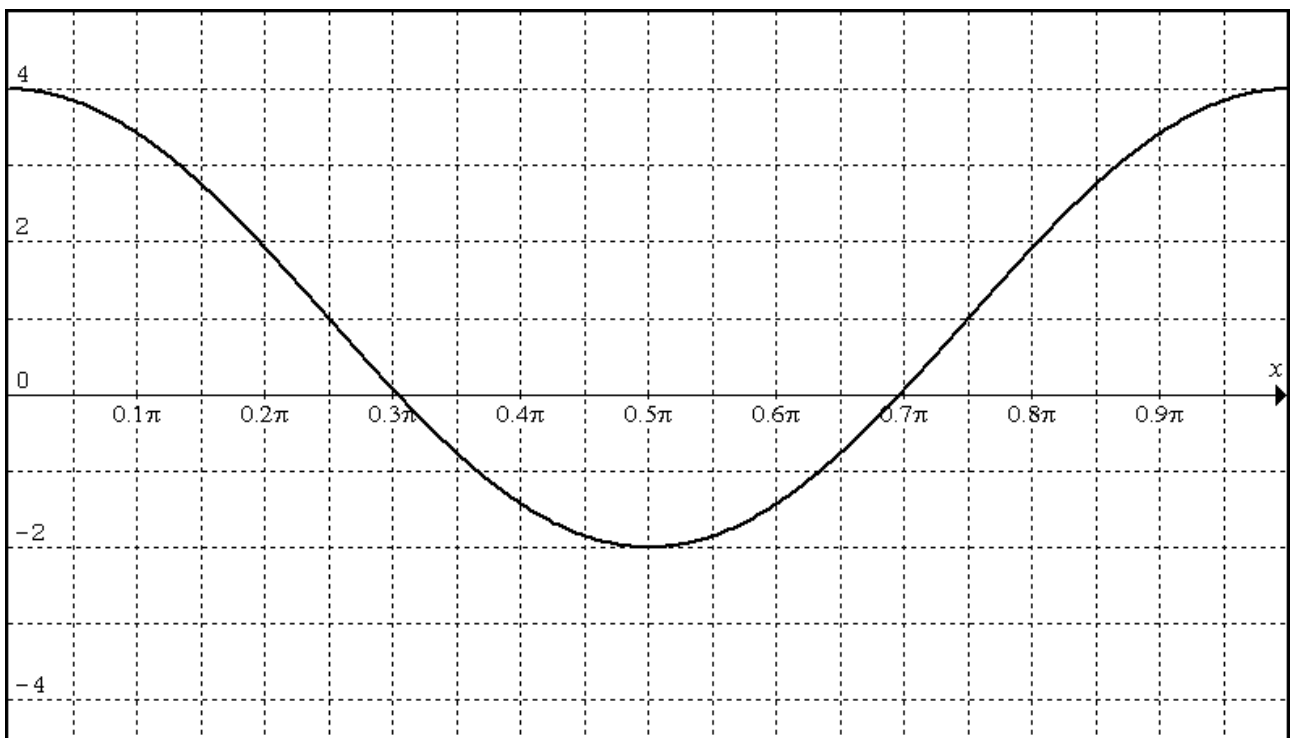
Step 2: Sketch $y = 3 \sin 2\left(x + \frac{\pi}{4}\right)$



Step 3: Sketch $y = 3 \sin 2\left(x + \frac{\pi}{4}\right) + 1$



Now we must only sketch the domain we want $(0, \pi)$



I have just enlarged the section above so we can see clearly what is happening in case you thought it looked a little different.

So now we can sketch these graphs lets us return to finding the solution of trigonometric equations

Say you are to solve the following equation: $\sin 2x = -\frac{1}{2}$ for $0 \leq x \leq 2\pi$

Let's look at what that means graphically.

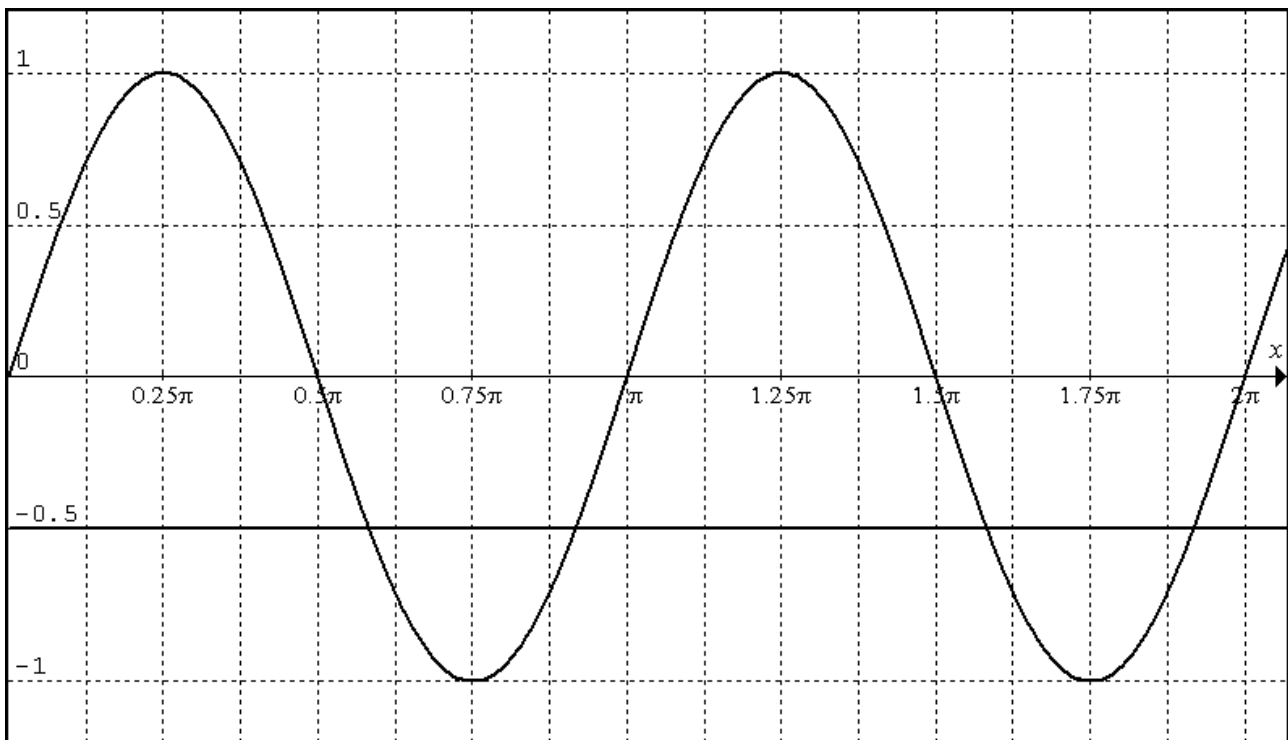
Sketch $y = \sin 2x$ and $y = -\frac{1}{2}$ on the same axis between $0 \leq x \leq 2\pi$

Solution:

Step 1: Sketch $y = \sin 2x$ for $0 \leq x \leq 2\pi$

Step 2: Sketch $y = -\frac{1}{2}$ for $0 \leq x \leq 2\pi$

Step 3: The intersect points are the solution!



Notice the solutions: There are 4 intersect points inside that domain: $x=1.83, 2.87, 4.97, 6.02$

But how do we find these solutions algebraically!

Follow the solution below.

(There are many ways of arriving at the answer below is just one of the ways)

Solve: $\sin 2x = -\frac{1}{2}$ for $0 \leq x \leq 2\pi$

Step 1

Firstly look at the fact that we have a negative answer which means that sin is only negative in quadrant 3 and quadrant 4.

Step 2

Next look at $\sin 2x$ there we will need to adjust the domain to be $0 \leq 2x \leq 4\pi$ and not $0 \leq x \leq 2\pi$

Step 3

The basic angle is $\frac{\pi}{6}$ since the inverse of $\sin x = \frac{1}{2}$

Step 4

$$2x = (\pi + \frac{\pi}{6}), (2\pi - \frac{\pi}{6})$$

$$2x = (\frac{7\pi}{6}) + (\frac{11\pi}{6}) + \text{repeats of these two angles every } \pm 2\pi$$

$$2x = (\frac{7\pi}{6}), (\frac{11\pi}{6}), (\frac{7\pi}{6}) + 2\pi, (\frac{11\pi}{6}) + 2\pi, (\frac{7\pi}{6}) + 4\pi, (\frac{11\pi}{6}) + 4\pi$$

$$2x = (\frac{7\pi}{6}), (\frac{11\pi}{6}), (\frac{19\pi}{6}), (\frac{23\pi}{6}), \text{outside the range others}$$

→ now divide all by 2

$$x = (\frac{7\pi}{12}), (\frac{11\pi}{12}), (\frac{19\pi}{12}), (\frac{23\pi}{12}),$$

Here we have the complete solution. It takes a few moments to understand by with practice we are able to solve algebraic trigonometric equation easily.

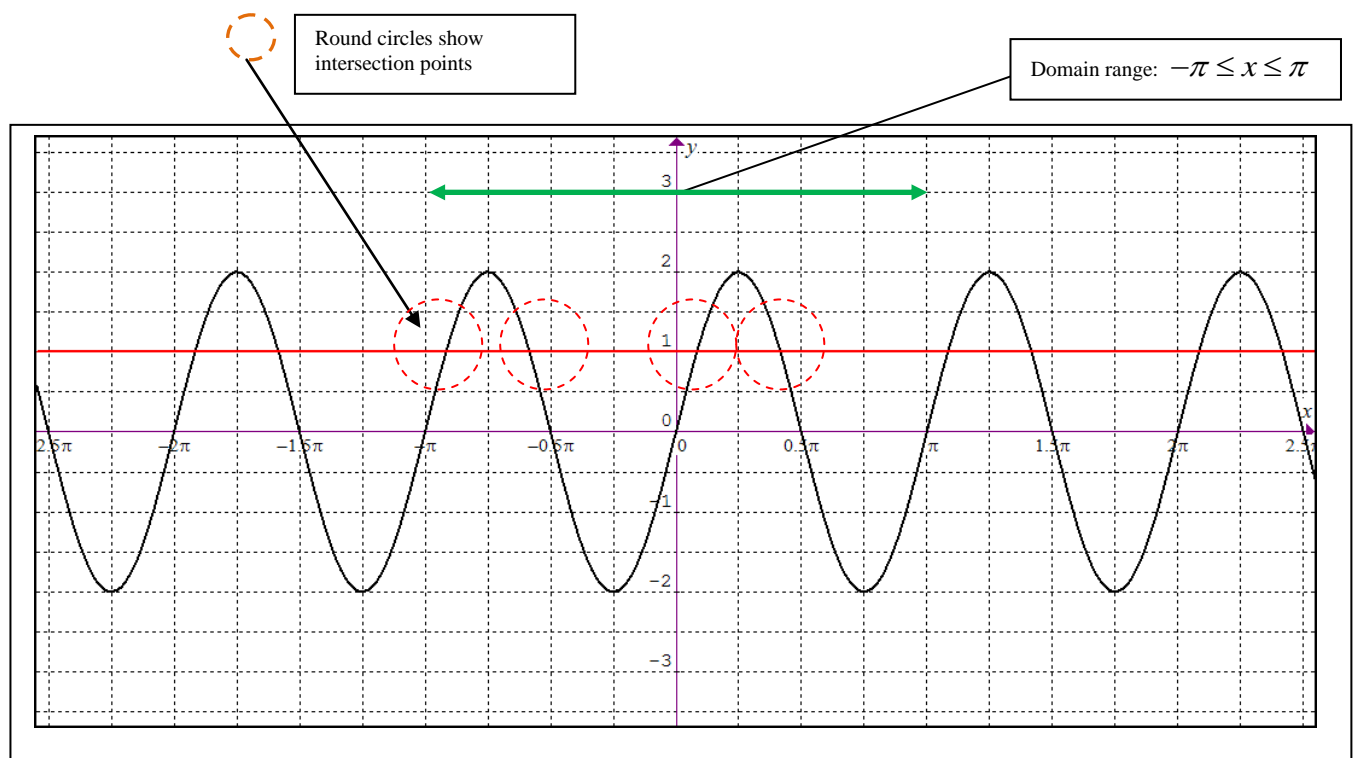
Solving Trigonometric Equations

Example: Solve $2\sin(2x) = 1$ for the following domain $-\pi \leq x \leq \pi$

Graphical Solution

Sketch the graph of $y = 2\sin(2x)$ and sketch the graph of $y = 1$

Once we have sketch the following two graphs, where they intercept is the required solutions



So we can see clearly that there will exist 4 solutions with the specified domain for this trigonometric equation. Of course we can use a graphics calculator and immediately find the required solutions approximately

$$x = -2.879, -1.8325, 0.26179, 1.30899$$

Now we will solve it using an algebraic method. To do this there are many different ways math's texts books use to solve, lets us use one of these.

First Step is to find the basic angle

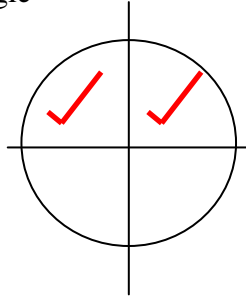
$$2\sin(2x) = 1$$

$$\sin(2x) = \frac{1}{2}$$

$$2x = \frac{\pi}{6} \text{ and } \frac{5\pi}{6}$$

$$2x = \frac{\pi}{6}$$

$$x = \frac{\pi}{12}$$



Notice that sine is positive in the first and second quadrant.

So since the basic angle is $\frac{\pi}{6}$, therefore the angle in the second quadrant will be $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$

$$2x = \frac{5\pi}{6}$$

And the other angle would be

$$x = \frac{5\pi}{12}$$

So these are the two angles namely, $\frac{\pi}{12}$ and $\frac{5\pi}{12}$

Now we have the basic angles

This trigonometric function repeats every period. Now to find the period of a sine function we simply divide by 2

Thus we obtain the following: $y = 2\sin(2x)$

$$\text{Period is } \frac{2\pi}{2} = \pi$$

So the answers will be $\pm \pi$ (it will repeat every period essentially), $\pm \pi$, $\pm 2\pi$, $\pm 3\pi$, $\pm 4\pi$ and so on.

So lets us find all the solutions

Here are the positives of the period

$$x = \frac{\pi}{12}, \pi + \frac{\pi}{12}, 2\pi + \frac{\pi}{12}, 3\pi + \frac{\pi}{12}$$

$$x = \frac{\pi}{12}, \frac{13\pi}{12}, \frac{25\pi}{12}, \frac{37\pi}{12}$$

We can have the negatives of the period

$$x = \frac{\pi}{12}, \frac{\pi}{12} - \pi, \frac{\pi}{12} - 2\pi,$$

$$x = \frac{\pi}{12}, \frac{-11\pi}{12}, \frac{-23\pi}{12}$$

Now we repeat for the other angle the same method

$$x = \frac{5\pi}{12}, \pi + \frac{5\pi}{12}, 2\pi + \frac{5\pi}{12}, 3\pi + \frac{5\pi}{12}$$

$$x = \frac{5\pi}{12}, \frac{17\pi}{12}, \frac{29\pi}{12}, \frac{41\pi}{12}$$

We can have the negatives of the period

$$x = \frac{5\pi}{12}, \frac{5\pi}{12} - \pi, \frac{5\pi}{12} - 2\pi,$$

$$x = \frac{5\pi}{12}, \frac{-7\pi}{12}, \frac{-19\pi}{12}$$

Now notice the domain that we must find the answers in, it is $-\pi \leq x \leq \pi$



The ones circles are within the domain given and so they can be included in the solution set.

So the answers are finally the following $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{-7\pi}{12}, \frac{-11\pi}{12}$ which of course correspond to the following answers from the graphical solutions 0.261799, 1.30899, -1.8325, -2.87979

Remember the answers must be given in the domain $-\pi \leq x \leq \pi$

This corresponds to -3.1415 to 3.1415 the domain set by the question initially.

Is there an easier way to work this out? Yes many different ways ranging from graphical solutions to shorter methods of solving trigonometric equations.

Example 1

$$2 \sin 2\theta = \sqrt{3}, \text{ for } 0 \leq \theta \leq 2\pi$$

Step 1: Find the quadrant in which sin is positive: Quadrant 1 and Quadrant 2

Step 2: Look at the domain, notice it is 2θ so we will have to find solutions between $0 \leq 2\theta \leq 4\pi$

Step 3: Now we attempt to find at least one of the primary solution

$$2 \sin 2\theta = \sqrt{3}$$

$$\rightarrow \sin 2\theta = \frac{\sqrt{3}}{2}$$

$$\rightarrow 2\theta = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$\rightarrow 2\theta = \frac{\pi}{3} \text{ (basic solution in quadrant 1)}$$

Now remember we have positive sin in Quadrant 1 and Quadrant 2

$$\rightarrow 2\theta = \frac{\pi}{3} \text{ and } \pi - \frac{\pi}{3}$$

$$\rightarrow 2\theta = \frac{\pi}{3} \text{ and } \frac{2\pi}{3} \text{ (plus repeats every } \pm 2\pi)$$

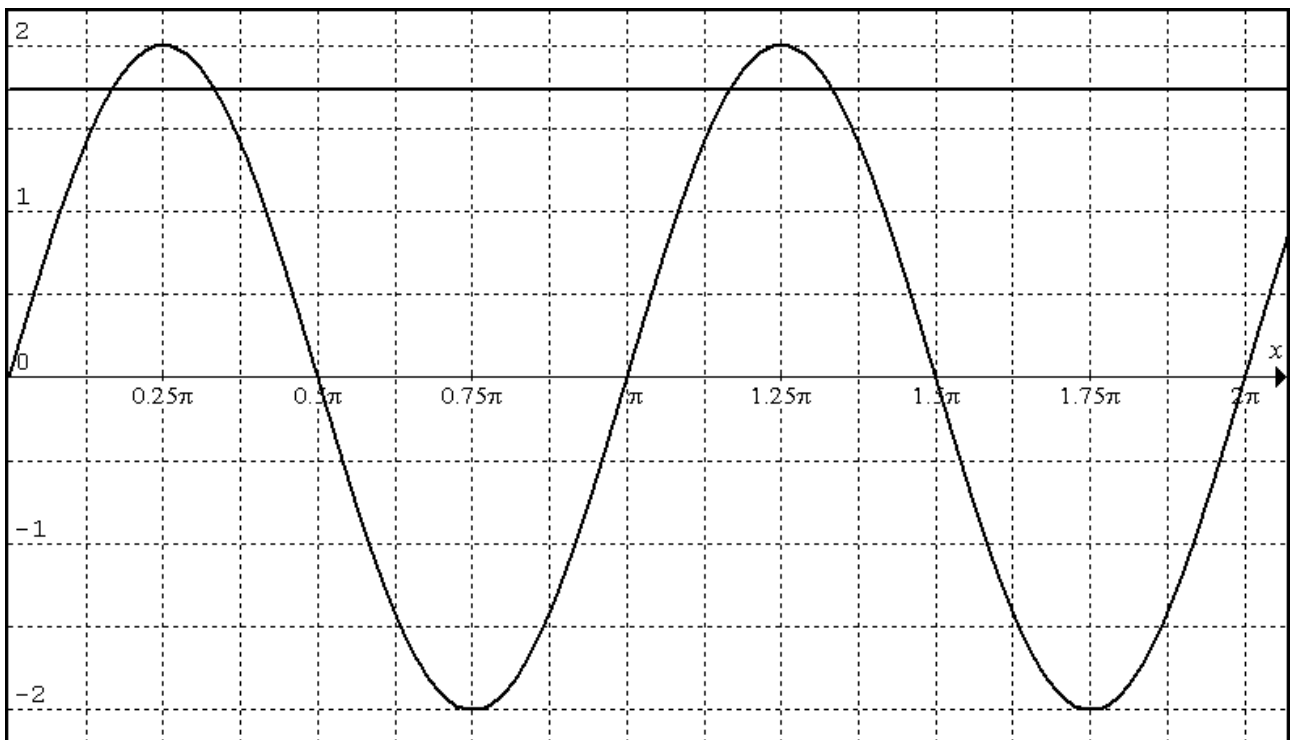
$$\rightarrow 2\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3} + 2\pi, \frac{2\pi}{3} + 2\pi, \text{ other solutions outside range}$$

$$\rightarrow 2\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$$

divide all by 2

$$\theta = \frac{\pi}{6}, \frac{2\pi}{6}, \frac{7\pi}{6}, \frac{8\pi}{6}$$

Of course we can check our solutions by sketching the two graphs of $y = 2 \sin 2\theta$ and $y = \sqrt{3}$



PROBLEM SHEET CHAPTER 6A,B,C,D,E,F,G,J,K,L,N,

How to sketch trigonometric graph by hand and without a calculator?

2006 Exam-1

For the function $f : [-\pi, \pi] \rightarrow R, f(x) = 5 \cos\left(2\left(x + \frac{\pi}{3}\right)\right)$

Part-1

Write down the amplitude and the period of the function

Part-2

Sketch the graph of the function on the axes below. Label the axes intercepts with their coordinates. Make sure to label the endpoints of the graph with the coordinates.

Solution step by step

Part-1

Amplitude is the height of the graph above the x-axis. So it is obvious that the amplitude of the graph is 5

Period is how often the graph repeats itself. You need to be very careful when working out the period of trigonometric graphs as there are traps for the student who does not concentrate.

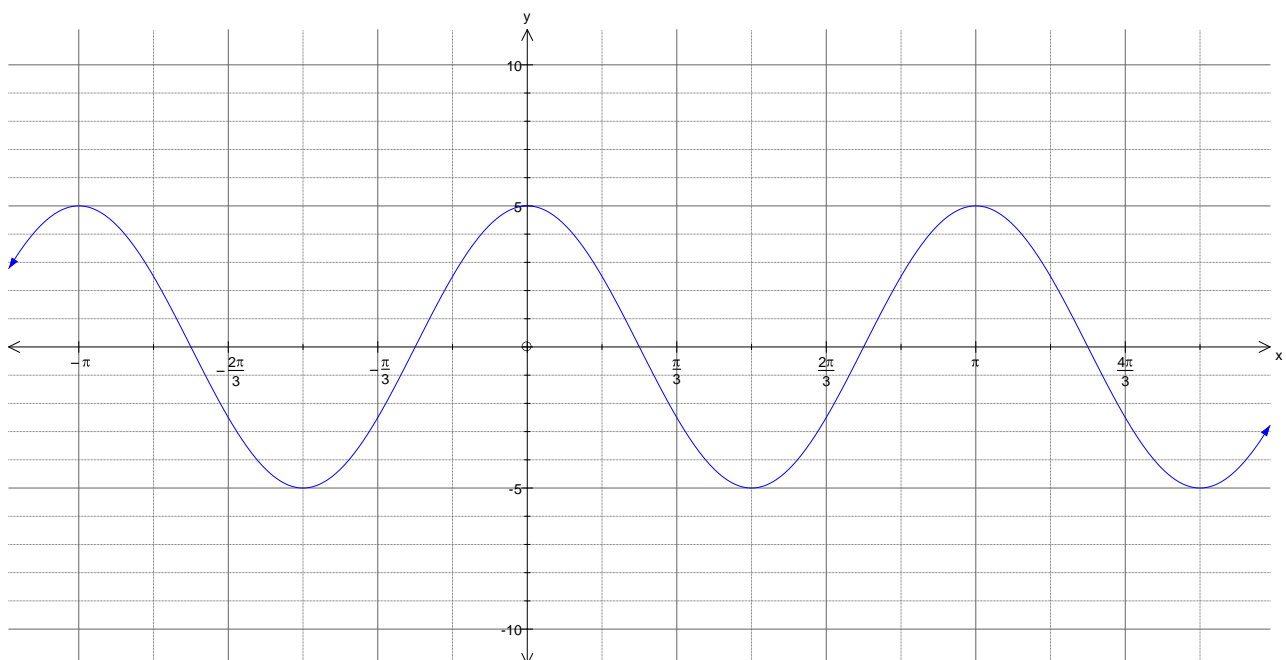
So let us examine what the traps are:

Equation	Period
$y = \cos nx$	$T = \frac{2\pi}{n}$
$y = \cos 2x$	$T = \frac{2\pi}{2}$
$f(x) = 5 \cos\left(2\left(x + \frac{\pi}{3}\right)\right)$	$T = \frac{2\pi}{2} = \pi$

Part-2-

Sketching graph is pretty difficult especially the cosine graphs.

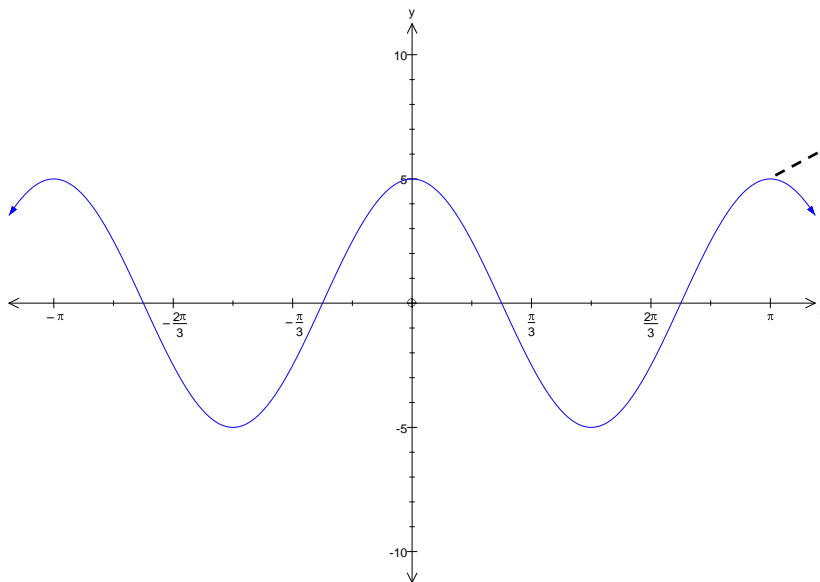
Step-1- Sketch the graph of $f(x) = 5 \cos(2x)$



Always draw a little bit more than the required domain so that you will be able to get all the points!

So you can see what happens when we have a horizontal translation of $+\frac{\pi}{3}$, the graph is moved to the left

So what we can do is pick points on the original graph and find the new values



So the point will be moved to the left
The new value will be given by the working
out the value of $\pi - \frac{\pi}{3}$
 $\rightarrow \frac{3\pi - \pi}{3} \rightarrow \frac{2\pi}{3}$
Now you repeat this for all important points

So picking out the main points we do the calculations and find the new points

Here is a table that shows the calculations. Obviously in an exam setting you will not go to this left but this example just highlights the method we can use to find the points

Old X-value	New X-value	
Original Point on the x-axis Starting from the right and moving left	Now we apply the transformation, so we subtract $-\frac{\pi}{3}$ from each x point	Now look at the domain remember it is only for $-\pi \leq x \leq \pi$
$\pi + \frac{\pi}{4} \rightarrow \frac{5\pi}{4}$	$\frac{5\pi}{4} - \frac{\pi}{3} = \frac{15\pi - 4\pi}{12} = \frac{11\pi}{12}$	$\left(\frac{11\pi}{12}, 0\right)$
π this is the end point	$\pi - \frac{\pi}{3} = \frac{3\pi - \pi}{3} = \frac{2\pi}{3}$	$\left(\pi, \frac{-5}{2}\right)$
$\frac{3\pi}{4}$ just a little explanation on this point if you cannot see how I got this point $\frac{\pi}{2} + \frac{\pi}{4} = \frac{4\pi + 2\pi}{8} = \frac{6\pi}{8} = \frac{3\pi}{4}$	$\frac{3\pi}{4} - \frac{\pi}{3} \rightarrow \frac{9\pi - 4\pi}{12} \rightarrow \frac{5\pi}{12}$	$\left(\frac{5\pi}{12}, 0\right)$

$\frac{\pi}{2}$	$\frac{\pi}{2} - \frac{\pi}{3} \rightarrow \frac{3\pi - 2\pi}{12} \rightarrow \frac{\pi}{12}$	$\left(\frac{\pi}{12}, 0\right)$
$\frac{\pi}{4}$	$\frac{\pi}{4} - \frac{\pi}{3} \rightarrow \frac{3\pi - 4\pi}{12} \rightarrow \frac{-\pi}{12}$	$\left(\frac{-\pi}{12}, 0\right)$
0,	$0 - \frac{\pi}{3} \rightarrow \frac{-\pi}{3}$	$\left(\frac{-\pi}{3}, 0\right)$
$-\frac{\pi}{4}$	$-\frac{\pi}{4} - \frac{\pi}{3} \rightarrow \frac{-3\pi - 4\pi}{12} \rightarrow \frac{-7\pi}{12}$	$\left(\frac{-7\pi}{12}, 0\right)$
$-\frac{\pi}{2}$	$-\frac{\pi}{2} - \frac{\pi}{3} \rightarrow \frac{-3\pi - 2\pi}{6} \rightarrow \frac{-5\pi}{6} \rightarrow \frac{-10\pi}{12}$	$\left(\frac{-10\pi}{12}, 0\right)$
$-\frac{3\pi}{4}$	$-\frac{3\pi}{4} - \frac{\pi}{3} \rightarrow \frac{-9\pi - 4\pi}{12} \rightarrow \frac{-13\pi}{12}$	Outside domain
$-\pi$	$-\pi - \frac{\pi}{3} \rightarrow \frac{-3\pi - \pi}{3} \rightarrow \frac{-4\pi}{3}$	Outside domain

Now fixing the end points

Endpoints we need to work out the value of y for the value of x	Put into the equation and use the ideas of trigonometry to find the value of y
$x = \pi$	$f(x) = 5 \cos\left(2\left(x + \frac{\pi}{3}\right)\right) \rightarrow 5 \cos\left(2\left(\pi + \frac{\pi}{3}\right)\right) \rightarrow 5 \cos\left(2\pi + \frac{2\pi}{3}\right) \rightarrow 5 \cos\left(\frac{2\pi}{3}\right) \rightarrow \frac{-5}{2}$
$x = -\pi$	$f(x) = 5 \cos\left(2\left(x + \frac{\pi}{3}\right)\right) \rightarrow 5 \cos\left(2\left(-\pi + \frac{\pi}{3}\right)\right) \rightarrow 5 \cos\left(-2\pi + \frac{2\pi}{3}\right) \rightarrow 5 \cos\left(\frac{2\pi}{3}\right) \rightarrow \frac{-5}{2}$

So we have found the endpoint values which are the following

$$\left(\pi, \frac{-5}{2}\right) \text{ and } \left(-\pi, \frac{-5}{2}\right)$$

Y-Intercept (You need to use the unit circle to work out the relationship here in simplified form)*

See bottom of this page for explanation

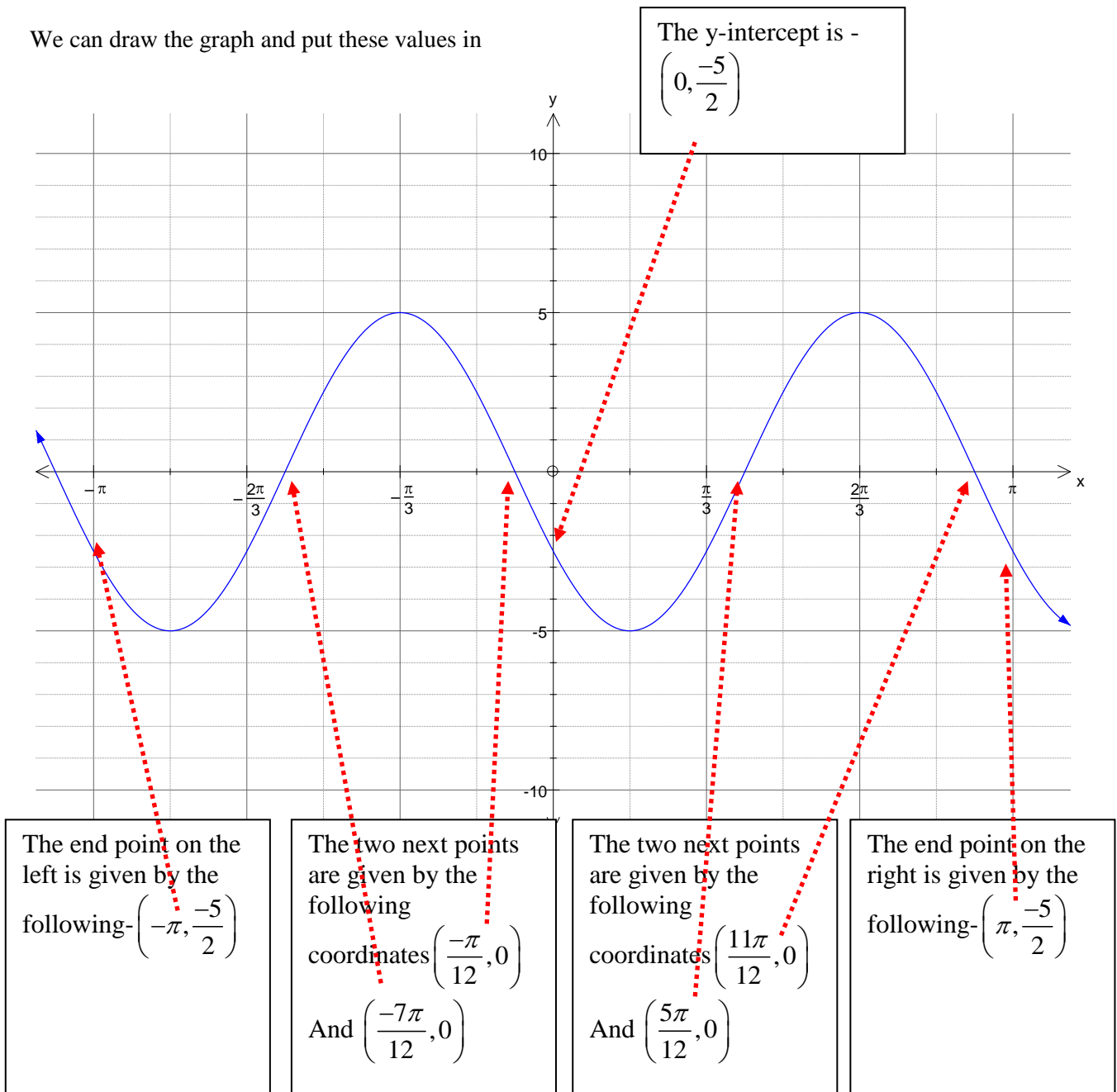
Also when $x = 0$ then

$$f(x) = 5 \cos\left(2\left(x + \frac{\pi}{3}\right)\right) \rightarrow 5 \cos\left(2\left(0 + \frac{\pi}{3}\right)\right) \rightarrow 5 \cos\left(+\frac{2\pi}{3}\right) \rightarrow 5 \cos\left(\frac{2\pi}{3}\right) \rightarrow \frac{-5}{2}$$

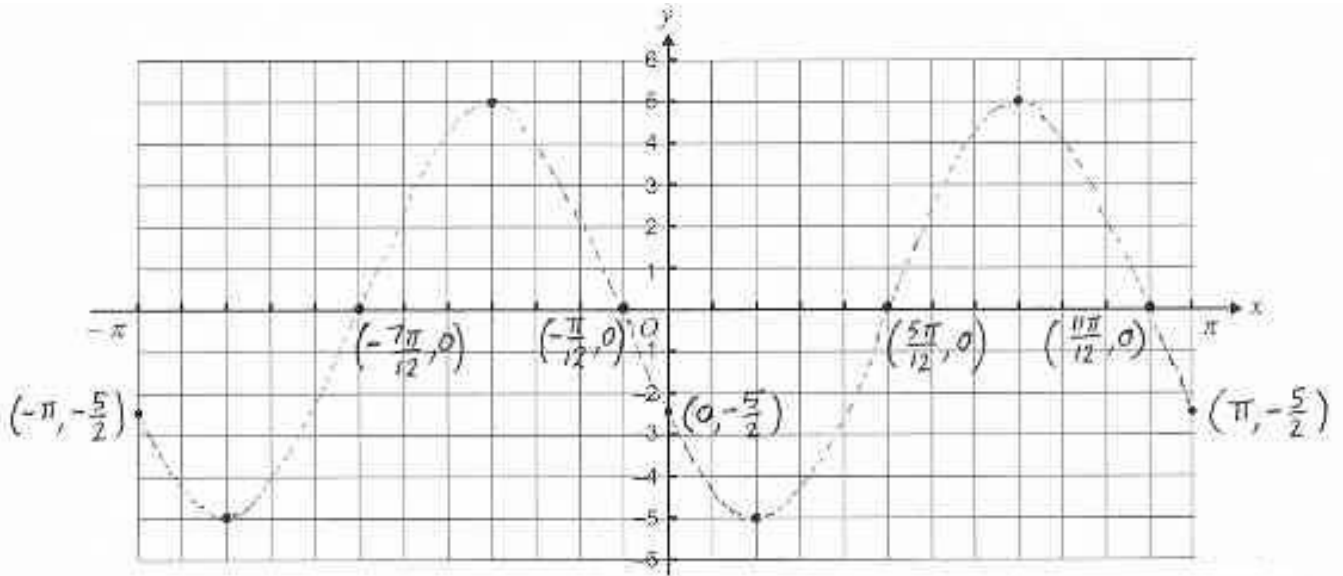
So the y intercept is then given the following coordinates

$$\left(0, \frac{-5}{2}\right)$$

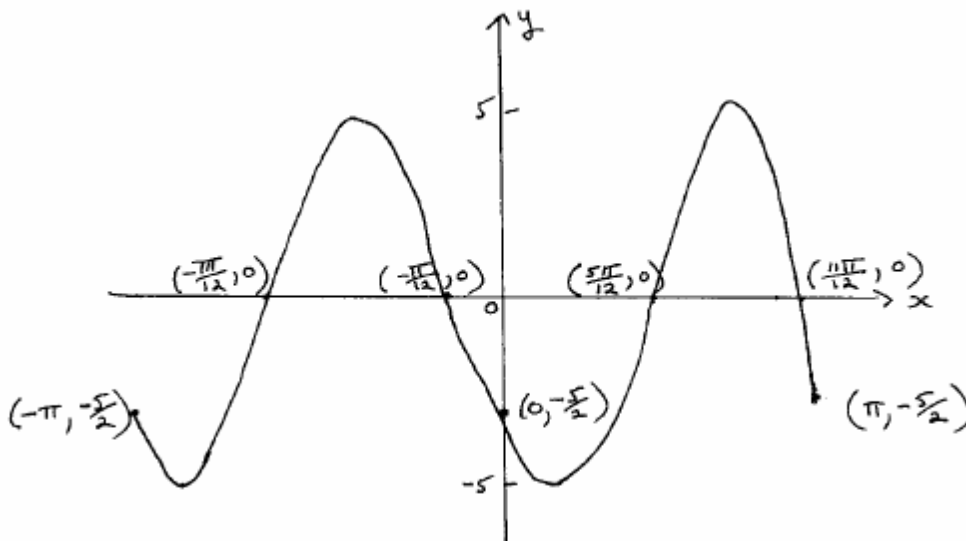
We can draw the graph and put these values in



Now the examiners were expecting this type of graph in the exam



Or a graph that looked like this



Working out the y intercept using the Unit circle

Remember if we add $+2\pi$ we arrive back to the beginning of the unit circle so then we get the following relationship

$$5 \cos\left(2\left(\pi + \frac{\pi}{3}\right)\right) \rightarrow 5 \cos\left(2\pi + \frac{2\pi}{3}\right) \rightarrow 5 \cos\left(\frac{2\pi}{3}\right) \rightarrow \frac{-5}{2}$$

You need to use the unit circle and the relationship to get this point. PLEASE REFER BACK TO NOTES

$\cos\left(\frac{2\pi}{3}\right)$ what is the angle of this ?

$$5 \cos\left(\pi - \frac{\pi}{3}\right) \rightarrow -5 \cos\left(\frac{\pi}{3}\right) \rightarrow \frac{-5}{2}$$

See which identity I used below?

And just in case you do not understand what I am mentioning here are the trigonometric identities that I am talking about

Symmetry

$\sin(-\theta) = -\sin \theta$	$\sin\left(\frac{\pi}{2} - \theta\right) = +\cos \theta$	$\sin(\pi - \theta) = +\sin \theta$
$\cos(-\theta) = +\cos \theta$	$\cos\left(\frac{\pi}{2} - \theta\right) = +\sin \theta$	$\cos(\pi - \theta) = -\cos \theta$
$\tan(-\theta) = -\tan \theta$	$\tan\left(\frac{\pi}{2} - \theta\right) = +\cot \theta$	$\tan(\pi - \theta) = -\tan \theta$

Here are some more

$\sin\left(\theta + \frac{\pi}{2}\right) = +\cos \theta$	$\sin(\theta + \pi) = -\sin \theta$	$\sin(\theta + 2\pi) = +\sin \theta$
$\cos\left(\theta + \frac{\pi}{2}\right) = -\sin \theta$	$\cos(\theta + \pi) = -\cos \theta$	$\cos(\theta + 2\pi) = +\cos \theta$
$\tan\left(\theta + \frac{\pi}{2}\right) = -\cot \theta$	$\tan(\theta + \pi) = +\tan \theta$	$\tan(\theta + 2\pi) = +\tan \theta$

Compliments of Wikipedia

http://en.wikipedia.org/wiki/List_of_trigonometric_identities

Sketching Graphs


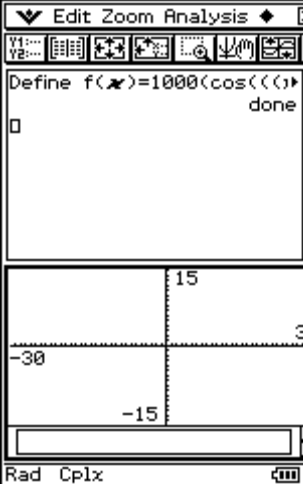
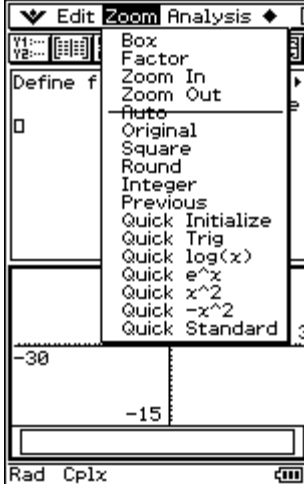
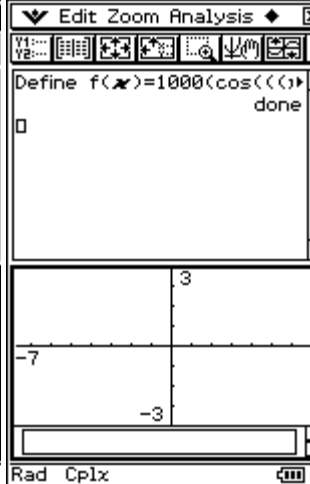
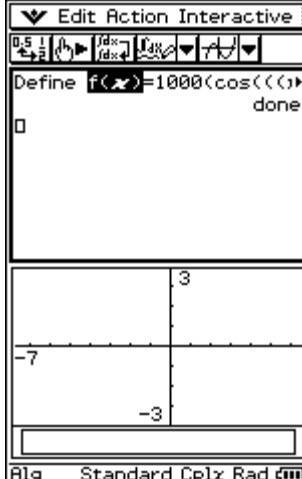
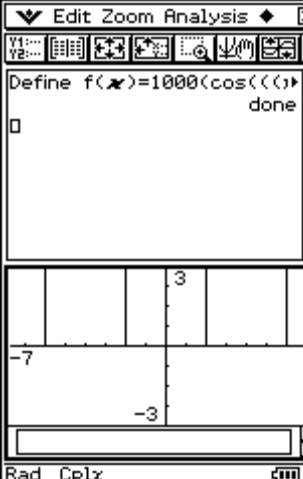
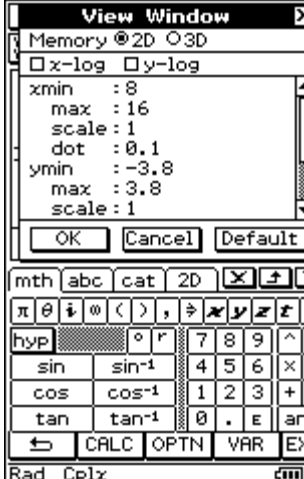
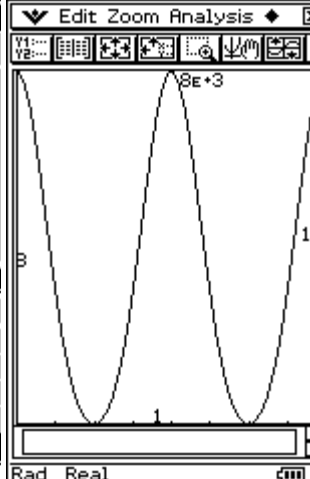

Sketch the following graph

$$f(x) = 1000 \left(\cos\left(\frac{\pi(x-8)}{2}\right) + 2 \right)^2 - 1000 \quad 8 \leq x \leq 16$$

The image shows four sequential screenshots of a TI-84 Plus calculator interface:

- Step-1:** The function $f(x) = 1000 \left(\cos\left(\frac{\pi(x-8)}{2}\right) + 2 \right)^2 - 1000$ is entered into the function editor.
- Step-2:** A context menu is shown with options like 'Cut', 'Copy', 'Paste', and 'Delete' over the function input.
- Step-3:** A menu is open showing various options such as 'Transformation', 'Calculation', and 'List-Create'.
- Step-4:** The 'Define' dialog box is shown with 'Func name: f', 'Variable/s: x', and 'Expression: 1000(cos((pi(x-8)/2)+2))^2-1000'.



Step-1-Put the function into Step-2-Select the function Step-3- Now go to Step-4-Now we define the

<p>the main window of calculator</p>	<p>,select edit and then select all to select the entire function</p>	<p>interactive and tap on define</p>	<p>equation by calling it</p>
			
<p>Step-5-Once you press OK from the previous step this is what you will see. Notice we have defined the equation with the symbol $f(x)$</p>	<p>Step-6- Tap the graph image which will open a graph window in the bottom half of the screen</p>	<p>Step-7-Tap Zoom Quick Initialize It set the screen</p>	<p>Step-8- This is what you will see when you have pressed zoom quick initialize</p>
			
<p>Step-9- Now select $f(x)$</p>	<p>Step-10-Drag it into the graph window and this is what you will see</p>	<p>Step-11- Press  and set the window setting. Use the domain information</p>	<p>Step-11- Press Zoom Auto and select full resize window and this is what you will get</p>
<p style="text-align: center;">Finding exact values for a particular function</p>			
<p>Finding the exact values of the above function how do we do it. For example how do we find $f(x) = 1250$, $8 \leq x \leq 16$</p>			

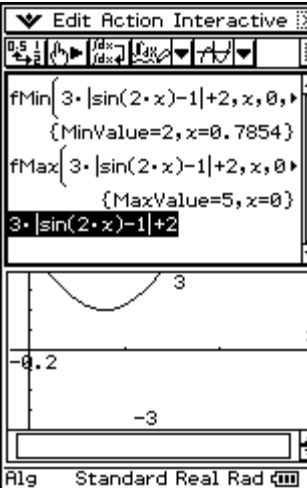

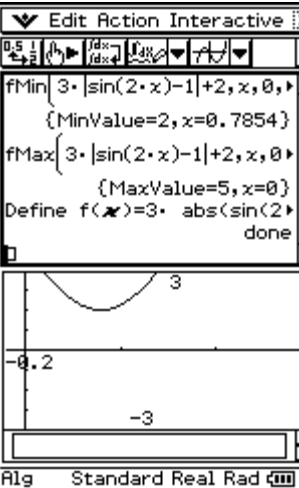
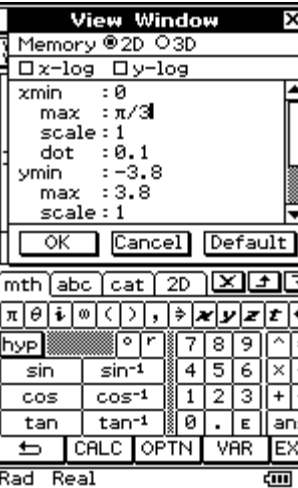
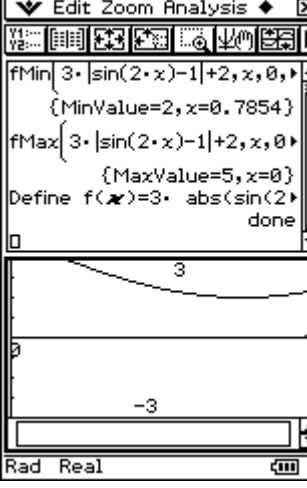
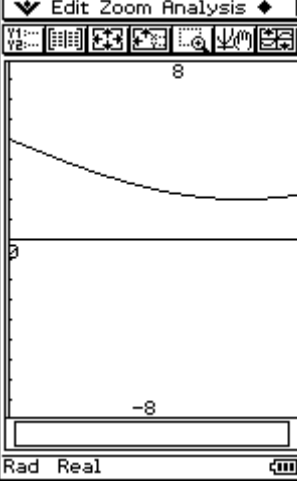
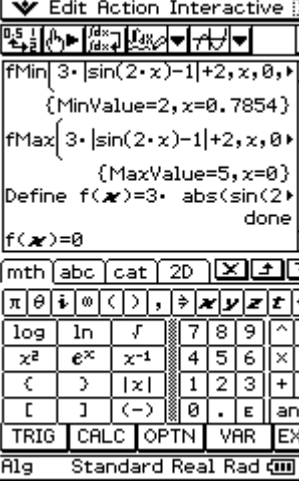
<p>Step -1- Set $f(x)=1250$</p>	<p>Step-2- Now highlight this expression by pressing Edit and select all</p>	<p>Step-3- Press interactive and then equation and then solve</p>	<p>Step-4- This is what you will see when you press solve Now press OK</p>
<p>Step-5- This is what you will see which does not make any sense does it?</p>	<p>Step-6- Put $8 \leq x \leq 16$ before the ,x</p>	<p>Step-7- see the graphical solution by typing 1250 and dragging them into graph window</p>	<p>Step-8- Make it full window</p>

Finding the minimum and maximum values in exact format

Find the range of the function $f : \left[0, \frac{\pi}{3}\right) \rightarrow R$ $f(x) = 3|\sin(2x) - 1| + 2$

<p>Make sure you set Alg, Standard, Real and Rad mode first</p>	<p>Now enter the equation very carefully Use 2D for the brackets</p>	<p>Put the equation in</p>	<p>Tap the graph  which will open the graph window in the bottom half</p>
<p>Now highlight the graph and drag it into the graph window</p>	<p>This is what you see. You might have to press ZOOM Quick Initialize</p>	<p>Now press the resize option</p>	<p>Press  to enter the domain information. Good to choose values just outside the value so you can see if the max or min occurs at the endpoints</p>

<p>Press OK gives the above</p>	<p>Select the function and go to interactive-calculation-fMin</p>	<p>This is the screen you will see before you press fMin</p>	<p>Now enter the start and the ending values</p>
<p>We have entered the start and the end values and we press OK</p>	<p>This is what we obtain once we press OK So the MinValue = 2 at x = 0.7854</p>	<p>Press the decimal to exact button once you have highlighted the value gives</p>	
			<p>CAUTION fMIN and fMAX will only return ONE VALUE for a minimum or maximum even if there are other values. So if the function repeats itself then we will need to use another method.</p>
<p>Highlight the function and go through the entire process</p>	<p>Put the values in and press OK</p>	<p>This is what we obtain</p>	<p>So the range is [2,5]</p>

 <p>Step-1-The other method-highlight the equation and we will define it</p>	 <p>Step-2-Define it by calling it f</p>	 <p>Step-3-Press OK</p>	 <p>Step-4-Adjust the domain again</p>
 <p>Step-5-Press OK</p>	 <p>Step-6-Press resize for graph</p>	 <p>Step-7-Now we want to find the derivative of f and set it to zero to find the values of the minimum</p>	<p>Highlight it and go</p>

Examples to consider

Solving circular functions algebraically

Algebraic solution of equations involving circular functions

It is prudent to consider the domain when solving an equation involving one or more circular functions.

In this section only equations involving only one of the circular functions are considered.

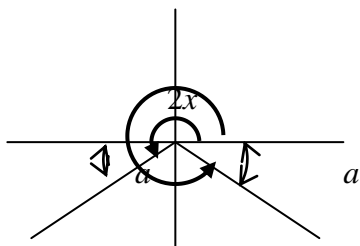
In solving such equations algebraically, always transpose an equation to make $\sin(kx)$, $\cos(kx)$ or $\tan(kx)$ the subject of the equation.

Example 1: Solve $3\sin(2x) = -2$, where $0 < x < 6$.

The domain is $0 < x < 6$, or $0 < 2x < 12$.

$$3\sin(2x) = -2, \quad \sin(2x) = -\frac{2}{3}$$

Since $\sin(2x)$ has a negative value, $-\frac{2}{3}$, then the 'angle' $2x$ must be in the third or fourth quadrant.



$$a = \sin^{-1}\left(\frac{2}{3}\right) = 0.7297 \text{ (Calculator)}$$

$$2x = \pi + 0.7297, 2\pi - 0.7297, 3\pi + 0.7297, 4\pi - 0.7297$$

$$2x = 3.8713, 5.5535, 10.1545, 11.8366$$

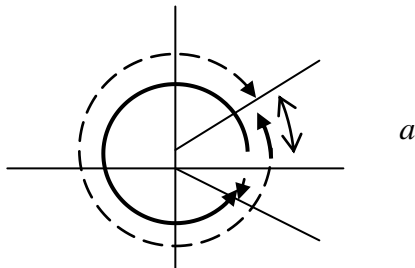
$$\therefore x = 1.9357, 2.7767, 5.0772, 5.9183$$

Example 2: Find the exact solution(s) of $2\sqrt{3}\cos\left(\frac{\theta}{2}\right) - 3 = 0$, where $-4\pi < \theta < 4\pi$.

The domain is $-4\pi < \theta < 4\pi$, or $-2\pi < \frac{\theta}{2} < 2\pi$.

$$2\sqrt{3}\cos\left(\frac{\theta}{2}\right) - 3 = 0, \cos\left(\frac{\theta}{2}\right) = \frac{\sqrt{3}}{2}.$$

Since $\cos\left(\frac{\theta}{2}\right)$ has a positive value, $\frac{\sqrt{3}}{2}$, \therefore the 'angle' $\frac{\theta}{2}$ must be in the first or fourth quadrant.



$$a = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

$$\frac{\theta}{2} = -2\pi + \frac{\pi}{6}, -\frac{\pi}{6}, \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$$

$$\therefore \frac{\theta}{2} = -\frac{11\pi}{6}, -\frac{\pi}{6}, \frac{\pi}{6}, \frac{11\pi}{6} \quad \therefore \theta = -\frac{11\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{11\pi}{3}$$

Example 3: Find the exact coordinates of the intersections of $y = 3\sin(2x) + 1$ and $y = \sqrt{3}\cos(2x) + 1$, where $-\pi < x < \pi$.

Solve the two equations simultaneously to find the intersections.

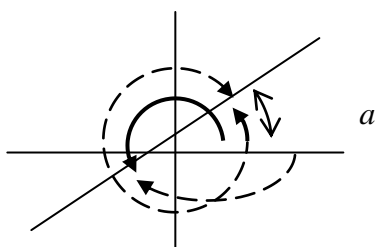
$$y = 3\sin(2x) + 1 \dots\dots\dots(1) \quad y = \sqrt{3}\cos(2x) + 1 \dots\dots\dots(2)$$

The domain is $-\pi < x < \pi$, or $-2\pi < 2x < 2\pi$.

Substitute eq.(1) in eq.(2), $3\sin(2x) + 1 = \sqrt{3}\cos(2x) + 1$,

$$\therefore 3\sin(2x) = \sqrt{3}\cos(2x), \frac{\sin(2x)}{\cos(2x)} = \frac{\sqrt{3}}{3}, \tan(2x) = \frac{1}{\sqrt{3}}.$$

Since $\tan(2x)$ has a positive value, $\frac{1}{\sqrt{3}}$, the 'angle' $2x$ must be in the first or third quadrant.



$$a = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$$2x = -2\pi + \frac{\pi}{6}, -\pi + \frac{\pi}{6}, \frac{\pi}{6}, \pi + \frac{\pi}{6},$$

$$\therefore 2x = -\frac{11\pi}{6}, -\frac{5\pi}{6}, \frac{\pi}{6}, \frac{7\pi}{6}, \therefore x = -\frac{11\pi}{12}, -\frac{5\pi}{12}, \frac{\pi}{12}, \frac{7\pi}{12}.$$

Substitute each of these (2x) values in eq.(1),

$$y = 3\sin\left(-\frac{11\pi}{6}\right) + 1 = 3\left(\frac{1}{2}\right) + 1 = \frac{5}{2},$$

$$y = 3\sin\left(-\frac{5\pi}{6}\right) + 1 = 3\left(-\frac{1}{2}\right) + 1 = -\frac{1}{2},$$

$$y = 3\sin\left(\frac{\pi}{6}\right) + 1 = 3\left(\frac{1}{2}\right) + 1 = \frac{5}{2}, \quad y = 3\sin\left(\frac{7\pi}{6}\right) + 1 = 3\left(-\frac{1}{2}\right) + 1 = -\frac{1}{2}. \text{ The coordinates are}$$

$$\left(-\frac{11\pi}{12}, \frac{5}{2}\right), \left(-\frac{5\pi}{12}, -\frac{1}{2}\right), \left(\frac{\pi}{12}, \frac{5}{2}\right) \text{ and } \left(\frac{7\pi}{12}, -\frac{1}{2}\right).$$

Example 4: Solve $\sin^2(x) = \sin(x)\cos(x)$ for x , where $0 \leq x \leq \frac{\pi}{2}$. Note: $\sin^2(x)$ is the proper notation for $(\sin(x))^2$.

Caution: Do not divide both sides of the equation by $\sin(x)$, because $\sin(x) = 0$ is a possibility. See below.

$$\sin^2(x) = \sin(x)\cos(x), \quad \sin^2(x) - \sin(x)\cos(x) = 0, \text{ common factor, } \sin(x)(\sin(x) - \cos(x)) = 0.$$

$$\therefore \text{either } \sin(x) = 0, \therefore x = 0,$$

$$\text{or } \sin(x) - \cos(x) = 0, \quad \frac{\sin(x) - \cos(x)}{\cos(x)} = \frac{0}{\cos(x)} \text{ for } \cos(x) \neq 0,$$

$$\therefore \tan(x) - 1 = 0, \quad \tan(x) = 1, \quad x = \frac{\pi}{4}.$$

The two possible solutions for x are $0, \frac{\pi}{4}$.

Example 5 Find t when $x = 10\sin\left(\frac{\pi t}{12}\right) - 12$ is a minimum, where $0 < t < 48$.

The domain is $0 < t < 48$, or $0 < \frac{\pi t}{12} < 4\pi$.

Minimum of x is -22 , it occurs when $\sin\left(\frac{\pi t}{12}\right) = -1$,

$$\therefore \frac{\pi t}{12} = \frac{3\pi}{2}, \frac{7\pi}{2}. \therefore t = 18, 42.$$

