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Calculus concepts

Year 12 Methods

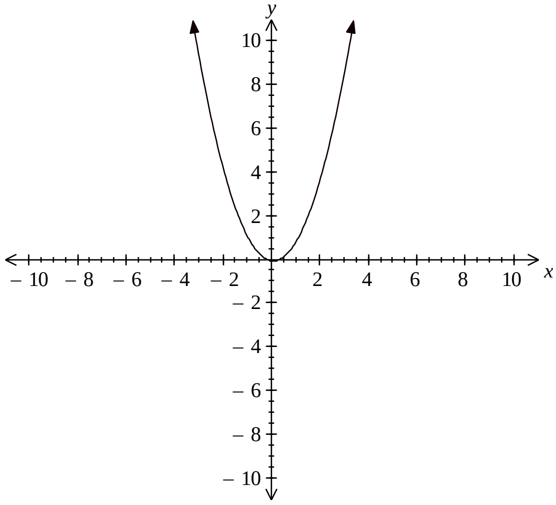
Quick concepts regarding calculus concepts

Calculus Concepts

Calculus is used in mathematics to solve problems such as determining the gradient of certain functions, determine the rate of change and solve maximum and minimum problems. It is an extremely important topic that has many applications.
THESE NOTES DO NOT COVER ALL THE CONCEPTS OF CALCULUS BUT JUST A BASIC INTROUCTION. THEY DO NOT SHOW HOW TO USE GRAPHICS CALCULATOR

Concept-1 Using first Principles

Calculus helps us determine the gradient (slope) of a curve at a particular point. Take the graph $y = x^2$



Say Point A has the following coordinates $(3,3^2) \rightarrow (3,9)$
And say Point B has the following coordinates $((3+h),(3+h)^2) \rightarrow (3+h,(3+h)^2)$

Now if we would like to determine the gradient between these two points we use the formula
$\text{gradient} = \frac{rise}{run}$
Now follow the logic and see what happens:

$$\text{Gradient} = \frac{rise}{run}$$
$$\overrightarrow{AB} = \frac{(3+h)^2 - (3^2)}{(3+h)-(3)}$$
$$\overrightarrow{AB} = \frac{(3+h)^2 - (9)}{(3+h)-(3)}$$
$$\overrightarrow{AB} = \frac{(9+6h+h^2)-(9)}{(3+h)-(3)}$$
$$\overrightarrow{AB} = \frac{6h+h^2}{h}$$
$$\overrightarrow{AB} = \frac{h(6+h)}{h}$$
$$\overrightarrow{AB} = 6+h$$

Now we have the gradient of the line joining these 2 points together being $\overrightarrow{AB} = 6 + h$

If now we made $h \to 0$ then $\overrightarrow{AB} = 6$

This method is the beginning of calculus. It is **called finding the derivative from first principals,** in other words finding the equation of the slope for a particular function, in the above the equation of the parabola.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This is the main equation we use when we want to find from first principals the gradient of a particular function.

Example 1

Find from first principals the derivative of $f(x) = 2x + 3$

Solution:

Step 1

First we find $f(x+h) \to 2(x+h) + 3 \to 2x + 2h + 3$

Step 2
Next step we follow the equation for first principals

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$f'(x) = \lim_{h \to 0} \frac{(2x+2h+3) - (2x+3)}{h}$$
$$f'(x) = \lim_{h \to 0} \frac{2h}{h}$$
$$f'(x) = \lim_{h \to 0} 2$$
Now we let $h \to 0$, but since there is no h
$$f'(x) = 2$$

Example 2

Find the derivative of $f(x) = x^3$ from 1^{st} principals.

Step 1: Find $f(x+h)$

$$f(x+h) =$$
$$\to = (x+h)^3$$
$$\to = (x+h)(x+h)(x+h)$$
$$\to = (x^2 + 2xh + h^2)(x+h)$$
$$\to = (x^3 + 3x^2h + xh^2 + h^3)$$

Notice how I left the brackets in to help me with the next step

Step 2: Put it into the general equation and work out slowly

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$f'(x) = \lim_{h \to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - (x^3)}{h}$$
$$f'(x) = \lim_{h \to 0} \frac{(3x^2h + 3xh^2 + h^3)}{h}$$
$$f'(x) = \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h}$$
$$f'(x) = \lim_{h \to 0} (3x^2 + 3xh + h^2)$$

Now we let $h \to 0$

$$f'(x) = 3x^2$$

Skill Builder

Question	Using First Principles find the derivate of the following functions	Answers
1	$f(x) = 3x + 2$	$f'(x) = 3$
2	$f(x) = 5x - 2$	$f'(x) = 5$
3	$f(x) = 2x^2$	$f'(x) = 4x$
4	$f(x) = 5x - x^2$	$f'(x) = 5 - 2x$
5	$f(x) = x^3 - x^2$	$f'(x) = 3x^2 - 2x$

Concept 2- Finding the derivate of functions faster

There is an easier way to find the derivative of the function above and that is by using the following rule:

If $f(x) = x^n$, where n is a positive or negative number then
$f'(x) = nx^{(n-1)}$

If the function starts with $f(x)$ then the derivate can be shown as $f'(x)$ whereas if the function starts with y then the derivate is shown as $\frac{dy}{dx}$

Example 3

Find the derivate for the following:

$$f(x) = 3x^4 + 5x^2$$
$$\rightarrow f'(x) = (3*4)x^{(4-1)} + (5*2)x^{(2-1)}$$
$$\rightarrow f'(x) = 12x^3 + 10x^1$$

Skill builder

Question	Using the shortcut method find the derivative	Answers
1	$f(x) = 3x + 2$	$f'(x) = 3$
2	$f(x) = 5x - 2$	$f'(x) = 5$
3	$f(x) = 2x^2$	$f'(x) = 4x$
4	$f(x) = 5x - x^2$	$f'(x) = 5 - 2x$
5	$f(x) = x^3 - x^2$	$f'(x) = 3x^2 - 2x$
6	$f(x) = 4x^3 - 3x^2 - 4$	$f'(x) = 12x^2 - 6x$
7	$f(x) = 2 - x^2$	$f'(x) = -2x$
8	$f(x) = \frac{4x^3}{5} - 7x^2$	$f'(x) = \frac{12x^2}{5} - 14x$
9	$f(x) = x^3 - x^2 - x^7$	$f'(x) = 3x^2 - 2x - 7x^6$
10	$f(x) = 2$	$f'(x) = 0$

Concept-3- Finding the gradient at a particular point of a function

So what if we are asked to find the gradient at a particular point, let's say x = 3 of the function $f(x) = 2x^2$, how do we go about it?

Answer

Step 1	Equation
Write down the equation	$f(x) = 2x^2$
Take the derivative using the short cut	$f'(x) = 4x$
Substitute x = 3 into the derivate to find the value of the gradient at that particular point	$f'(3) = 4(3) = 12$
So the gradient at x = 3 is 12	

Skill builder

Question	Using the shortcut method find the derivative	Find the gradient at x = 2
1	$f(x) = 3x + 2$	$f'(x) = 3$
2	$f(x) = 5x - 2$	$f'(2) = 10$
3	$f(x) = 2x^2$	$f'(x) = 8$
4	$f(x) = 5x - x^2$	$f'(x) = 1$
5	$f(x) = x^3 - x^2$	$f'(x) = 8$
6	$f(x) = 4x^3 - 3x^2 - 4$	$f'(x) = 36$

Concept-4- Recognizing how to rewrite certain equations into an easier form so you can take the derivative.

Normal form	Easier to use form	Derivative
$y=\sqrt{x}$	$y=x^{\frac{1}{2}}$	$\frac{dy}{dx}=\frac{1}{2}x^{\frac{-1}{2}}$
$y=\sqrt[3]{x}$	$y=x^{\frac{1}{3}}$	$\frac{dy}{dx}=\frac{1}{3}x^{\frac{-2}{3}}$
$y=\sqrt[3]{x^2}$	$y=x^{\frac{2}{3}}$	$\frac{dy}{dx}=\frac{2}{3}x^{\frac{-1}{3}}$

Example 4

Problem: Find the gradient at (1, 2) for the graph $f(x)=x^2+\frac{1}{x}$

Solution:

We must find the derivate of the function firstly and then substitute x = 1 into the derivate to find the gradient. But here we need to be careful!

Step 1: Put the function into an easier form

$$f(x)=x^2+\frac{1}{x}$$
$$f(x)=x^2+x^{-1}$$
$$\rightarrow f'(x)=2x-x^{(-1-1)}$$
$$\rightarrow f'(x)=2x-x^{-2}$$

Now substitute x=1

$$f'(1)=2(1)-(1)^{-2}$$
$$f'(1)=2-1$$
$$f'(1)=1$$

Skill Builder

1	Find the gradient at x= 2 for the graph $f(x)=x^2+\frac{1}{x}$
2	Find the gradient at $x=-3$ for the graph $f(x)=x^2+3x$

Concept –5- Using the chain rule

$y = u^n$

$\rightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

- Used to differentiate "composite functions" in other words where one function is inside as it were of another function
- Example of composite functions: $y = (3x^2 + x)^4$

How do we use the chain rule? Let's look at a step by step example

find $\frac{dy}{dx}$ of $y = (3x^2 + x)^4$	Always call the inside function " u"	
$y = (3x^2 + x)^4$	$u = 3x^2 + x$	$y = u^4$
	$\frac{du}{dx} = 6x + 1$	$\frac{dy}{du} = 4u^3$
$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ $\frac{dy}{dx} = (4u^3) \times (6x + 1)$ $\frac{dy}{dx} = (4(3x^2 + x)^3) \times (6x + 1)$		

When doing these types of questions you will need to be extremely careful and do not forget the brackets and set it out as above.

Find $\frac{dy}{dx}$ of $y = (2x^3 - 2x^5)^4$	Always call the inside function " u"	
	$u = (2x^3 - 2x^5)$	$y = u^4$
	$\frac{du}{dx} = 6x^2 - 10x^4$	$\frac{dy}{du} = 4u^3$
$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ $\frac{dy}{dx} = 4u^3 \times (6x^2 - 10x^4)$ $\frac{dy}{dx} = 4(2x^3 - 2x^5)^3 (6x^2 - 10x^4)$		

Skill Builder

Question	Using the chain rule find the derivative	Answers
1	$f(x)=\left(3x+2\right)^4$	
2	$f(x)=\left(5x-2\right)^{-3}$	
3	$f(x)=\left(2x^2\right)^5$	
4	$f(x)=\left(5x-x^2\right)^3$	
5	$f(x)=\left(x^3-x^2\right)^8$	
7	$f(x)=2-x^2$	

Concept-6- Using the Product Rule

- Used to differentiate two functions that are multiplied together
- $y=u\times v$ where u and v are functions
- Watch out sometimes you will have to use the chain rule along with the product rule so things can get a little long and confusing…

$y=uv$

- $\rightarrow \frac{dy}{dx}=u\bullet\frac{dv}{dx}+v\bullet\frac{du}{dx}$

Worked out example

Find $\frac{dy}{dx}$ of $y=(x^3+1)(x^2+3x+2)$	Always call the inside function " u"	Call the other function "v'
	$u=x^3+1$	$v=(x^2+3x+2)$
	$\frac{du}{dx}=3x^2$	$\frac{dv}{dx}=2x+3$
$\rightarrow \frac{dy}{dx}=u\bullet\frac{dv}{dx}+v\bullet\frac{du}{dx}$ $\rightarrow \frac{dy}{dx}=(x^3+1)(2x+3)+(x^2+3x+2)(3x^2)$ $\rightarrow \frac{dy}{dx}=(x^3+1)(2x+3)+(x^2+3x+2)(3x^2)$		

We can obviously simplify the above answer and I will leave it up to you to obtain a simpler answer.

Concept –7- Combination questions involving chain rule and product rule

Find $\frac{dy}{dx}$ of $y = x^3(3x-5)^4$			
Step 1- Use Product rule	$u = x^3$	$v = (3x-5)^4$	
$y = uv$ $\rightarrow \frac{dy}{dx} = u \bullet \frac{dv}{dx} + v \bullet \frac{du}{dx}$ $\rightarrow \frac{dy}{dx} = u \bullet \frac{dv}{dx} + v \bullet \frac{du}{dx}$ $\rightarrow \frac{dy}{dx} = x^3(12(3x-5)^3) + (3x-5)^4(3x^2)$ $\rightarrow \frac{dy}{dx} = 12x^3(3x-5)^3) + (3x^2)(3x-5)^4$ $\rightarrow \frac{dy}{dx} = (3x-5)^3(12x^3 + (3x^2)(3x-5))$ $\rightarrow \frac{dy}{dx} = (3x-5)^3(21x^3 - 15x^2)$	$\frac{du}{dx} = 3x^2$	$\frac{dv}{dx} = ?$ Need to use the chain rule $\frac{dv}{dx} = 12(3x-5)^3$	Use on the chain rule on a separate part and you will find that the

Concept-8- Quotient Rule

- Used when one function is divided by another function
- Here if you are not careful of negatives you will definitely make a mistake

$$y = \frac{u}{v},\ \mathrm{v} \neq 0$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Worked out example

Find $\frac{dy}{dx}$ of $\frac{x^2+1}{2x+3}$	Always call the top function " u"	Call bottom function "v'
	Let $u = x^2 + 1$	$v = 2x + 3$

	$\frac{du}{dx}=2x$	$\frac{dv}{dx}=2$
$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$ $\frac{dy}{dx}=\frac{[2x(2x+3)]-[(2(x^2+1)]}{[2x+3]^2}$ $\frac{dy}{dx}=\frac{[4x^2+6x]-[2x^2+2]}{[2x+3]^2}$ $\frac{dy}{dx}=\frac{4x^2+6x-2x^2-2}{[2x+3]^2}$ $\frac{dy}{dx}=\frac{2x^2+6x-2}{[2x+3]^2}$		

Watch out with Negatives and do not forget the brackets

Concept-9- Derivatives of Exponential functions

Find $\frac{dy}{dx}$ of $y=e^{kx}$	Always call the top function " u"	Function becomes
	Let $u=kx$ so the function then becomes $\frac{du}{dx}=k$	$y=e^u$ Now the derivative of the exponential function is the exponential function
Now we use the chain rule $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$ Putting everything in we get the following $\frac{dy}{dx}=e^u\times k$ $\frac{dy}{dx}=ke^{kx}$		$\frac{dy}{dx}=e^u$

This rule above is only valid for exponential functions of base *e*

If you are asked to differentiate exponentials other than base *e* such as $y=2^x$ you then must use your calculator to work out the value at a certain point.

Example 1

If $f(x) = 3^x$ find $f'(1)$

Solution:
Use graphics calculator: nDeriv (3^x, x, 1) =3.2958

For composite exponential functions i.e. $y = e^{f(x)}$ use chain rule.

Examples to see how the rule is used for exponential functions

Problem	Worked out solution
Find $\frac{dy}{dx}$ of $y = e^{(3x^2+x)}$	Use the chain rule Let $u = 3x^2 + x$ $\rightarrow \frac{du}{dx} = 6x + 1$ Let $y = e^u$ $\frac{dy}{du} = e^u$ Now using the chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ $\frac{dy}{dx} = e^u \bullet (6x+1)$ $\frac{dy}{dx} = (6x+1)e^{3x^2+x}$

Skill Builder

Question	Using the chain rule find the derivative	Answers
1	$f(x) = e^{(3x+2)}$	
2	$f(x) = e^{(5x-2)}$	
3	$f(x) = e^{\left(2x^2\right)^5}$	
4	$f(x) = e^{\left(5x-x^2\right)}$	
5	$f(x) = e^{\left(x^3-x^2\right)}$	
7	$f(x) = e^{(2-x^2)}$	

Concept-10- Derivatives of logarithmic functions

Find $\frac{dy}{dx}$ of $y = \log(kx)$ Follow the steps on the right Normally with practice you will be able to do this far quicker by remembering the definition.	Always call the top function " u" $\rightarrow kx = e^y$ Now we divide it by k $\frac{kx}{k} = \frac{e^y}{k}$ $x = \frac{e^y}{k}$ Now we take the derivative $\frac{dx}{dy} = \frac{e^y}{k}$ But $kx = e^y$ so $\frac{dx}{dy} = \frac{kx}{k}$ $\frac{dx}{dy} = x$ So flipping the derivative over (not supposed to do this) $\frac{dy}{dx} = \frac{1}{x}$

If asked to find the derivative of a logarithmic function other than base e must use the graphics calculator (Like the exponential case)

Find $\frac{dy}{dx}$ of $\log_e(3x^2 + x)$	Use the chain rule $u = 3x^2 + x$ $\rightarrow \frac{du}{dx} = 6x + 1$	$y = \log_e u$ $\frac{dy}{du} = \frac{1}{u}$
Chain rule application now		

$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$ $\rightarrow=\frac{1}{u}\times(6x+1)$ $\rightarrow=\frac{6x+1}{3x^2+x}$		

Skill Builder

Question	Using the chain rule find the derivative	Answers
1	$f(x)=\log_e(3x+1)$	
2	$f(x)=\log_e(4x^3)$	
3	$f(x)=\log_e(5x-x^3)$	
4	$f(x)=4x+\log_e(3x+1)$	
5	$f(x)=x^3\log_e(3x^2)$	
7	$f(x)=\log_e(1-x)$	

Concept-11- Derivatives of trigonometric functions

$y=\sin kx$	$\frac{dy}{dx}=k\cos kx$
$y=\cos kx$	$\frac{dy}{dx}=-k\sin kx$
$y=\tan kx$	$\frac{dy}{dx}=k\sec^2 kx$

Sometimes we can once again need to use the chain rule when tackling questions in trigonometric functions

$y=\sin(f(x))$	$\frac{dy}{dx}=f'(x)\cos(f(x))$
$y=\cos(f(x))$	$\frac{dy}{dx}=-f'(x)\sin(f(x))$
$y=\tan(f(x))$	$\frac{dy}{dx}=f'(x)\sec^2(f(x))$

Trigonometric functions can only be differentiated if the angle is in ***RADIANS!***

If x is in degrees then we should change it to radians.

Find $\frac{dy}{dx}$ if $y = \sin 2x^{\circ}$	Solution: Convert degrees to radians $2x^{\circ} \rightarrow \frac{2\pi x}{180}$ $y = \sin(\frac{2\pi x}{180})$ $\rightarrow \frac{dy}{dx} = \frac{2\pi}{180}\cos(\frac{2\pi x}{180})$	
Now we can convert back to degree form	$\frac{dy}{dx} = \frac{2\pi}{180}\cos 2x^{\circ}$	

Another Example

Find $\frac{dy}{dx}$ of $y = \sin(3x^2 + x)$	Solution Recognize we must use the chain rule here Let $u = 3x^2 + x$ $\frac{du}{dx} = 6x + 1$	Now we have $y = \sin u$ $\frac{dy}{du} = \cos u$
Chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ $\rightarrow = \cos u \times (6x+1)$ $\rightarrow = (6x+1)\cos(3x^2 + x)$		

WATCH OUT IF YOU ARE ASKED TO TAKE THE DERIVATIVE OF SOMETHING THAT LOOKS LIKE THIS $y = \sin^2 x \rightarrow y = (\sin x)^2$

Find $\frac{dy}{dx}$ of $y = \sin^2 x$		
First recognize that $y = \sin^2 x$ $\rightarrow y = (\sin x)^2$	Using the chain rule let u=$\sin x$ $\frac{du}{dx} = \cos x$	$y = u^2$ $\frac{dy}{du} = 2u$
Now we apply the chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ $\frac{dy}{dx} = 2u(\cos x)$ $\frac{dy}{dx} = 2\sin x \cos x$		

Concept-12-Combinations of the various rules

We can have questions that will force us to use combinations of the various rules such as the chain rule along with the product rule. It is important in these instances to set out the question neatly and proceed methodically in solving these questions

Example

Find $\frac{dy}{dx}$ of $y = x^3 \sin 4x$		
First recognize that this involves the product rule $y = \sin^2 x$ $\rightarrow y = (\sin x)^2$	Using the product rule let u=x^3 $\frac{du}{dx} = 3x^2$	$v = \sin 4x$ $\frac{dv}{dx} = 4\cos 4x$
Now we apply the chain rule $y = uv$ $\rightarrow \frac{dy}{dx} = u \bullet \frac{dv}{dx} + v \bullet \frac{du}{dx}$ Substituting all the information very		

carefully we get the following $\frac{dy}{dx}=\left(x^3\right)\left(4\cos 4x\right)+\left(\sin 4x\right)\left(3x^2\right)$		

Skill builder

Question	Using the chain rule find the derivative	Answers
1	$f(x)=x^3e^{(3x+2)}$	
2	$f(x)=(x^5+3x)e^{(5x)}$	
3	$f(x)=\frac{e^{\left(2x^2\right)^5}}{x^3}$	
4	$f(x)=(x^3-3x)\log_e(3x)$	
5	$f(x)=e^{\left(x^3-x^2\right)}$	

Concept –13- Application of Calculus-Equations of straight lines

To find the equation of a tangent to a curve $y=f(x)$ at point (x_1,y_1) use the equation

$$y-y_1=m(x-x_1)\text{ , where } m=f'(x)$$

To find the equation of a normal (the line perpendicular to the tangent) to the curve $y=f(x)$ at point (x_1,y_1) use the equation

$$y-y_1=-\frac{1}{m}(x-x_1)\text{ , where } m=f'(x)$$

Find the equation of the tangent to the curve $y=x^3+\frac{1}{2}x^2$ at point x=1		
Step 1- Find the coordinates of y at x = 1	$x=1\therefore\rightarrow y=(1)^3+\frac{1}{2}(1)^2$ $\rightarrow y=\frac{3}{2}$	
Step 2- Take the derivate of the curve	$y=x^3+\frac{1}{2}x^2$ $\rightarrow\frac{dy}{dx}=3x^2+x$	

Find the gradient at x = 1	$\frac{dy}{dx} = 3(1)^2 + 1$ $\frac{dy}{dx} = 4$	
The equation of the tangent is	$y - y_1 = m(x - x_1)$ $y - \frac{3}{2} = 4(x-1)$	

Skill Builder

Question		Answers
1	Find the equation of the tangent to the curve $y = x^3 + 3x$ at point x =1	
2	Find the equation of the normal to the curve $y = 3\log_e(2x+1) - 1$ at $x = 0$.	
3	Find the equation of the normal to the curve $y = x^3 + 3x$ at point x =1	
4	Find the equation of the normal to the curve $y = 6x^3 - 3x^2$ at point x =1	
5	The line $y = -2x + 1$ is a tangent to the parabola $y = x^2 - px + q$. Find the values of p and q.	

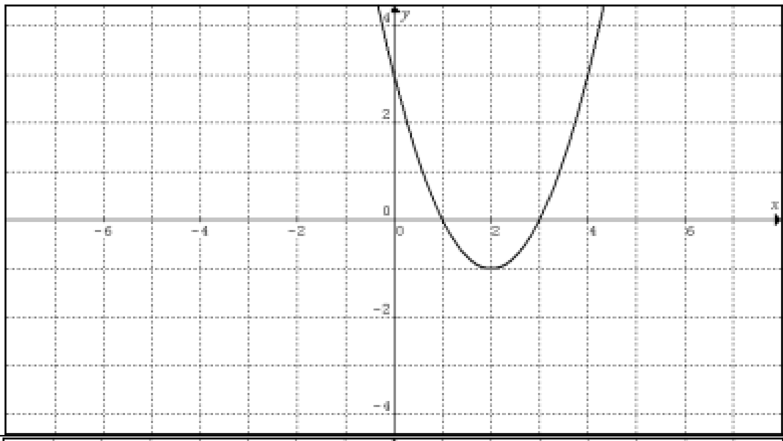
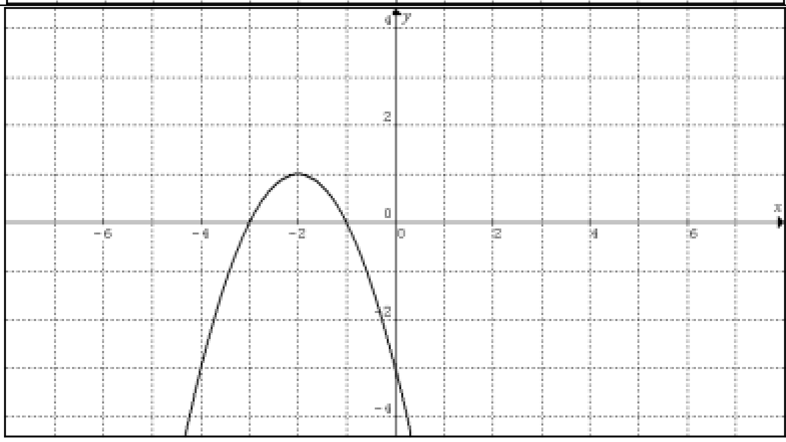
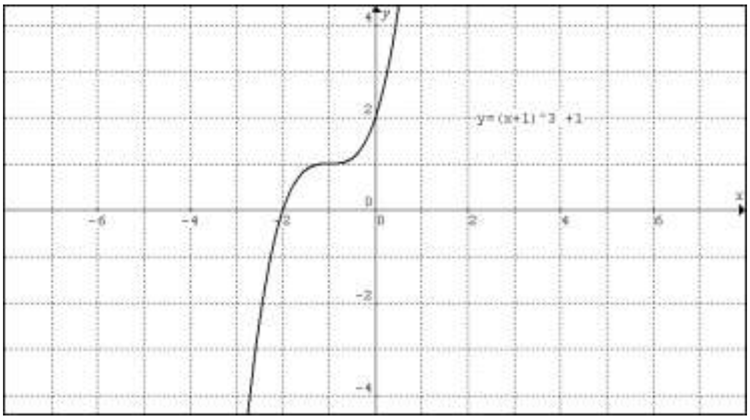
Concept –14- Application of Calculus-Stationary Points

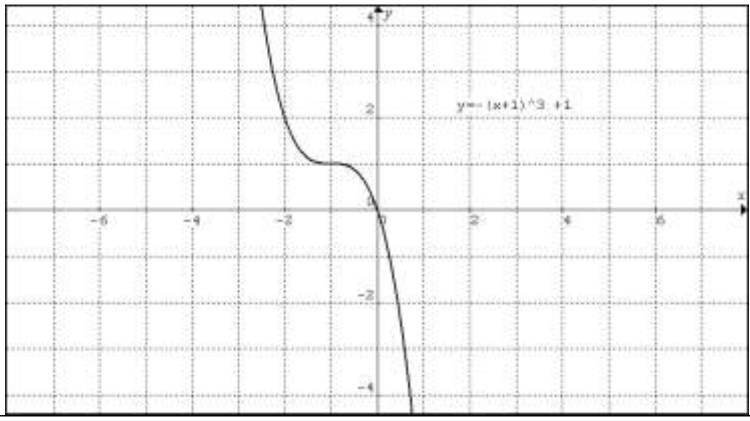
A Stationary point is any point on a graph $y = f(x)$ where $f'(x) = 0$

There are basically 3 types of stationary points

1. Local Minimum Point
2. Local Maximum Point
3. Stationary point of inflexion

Stationary Points	Graphs

Local Minimum Point Notice at x=2 we have a stationary point which is a minimum $x < a \quad f'(a) < 0$ $x = a \quad f'(a) = 0$ $x > a \quad f'(a) > 0$	
Notice here the following Maximum occurs at x=-2 $x < b \quad f'(b) > 0$ $x = b \quad f'(b) = 0$ $x > b \quad f'(b) < 0$	
Stationary Point of Inflexion Notice that something unique occurs at (-1, 1) this is a point of inflexion $x < c \quad f'(c) > 0$ $x = c \quad f'(c) = 0$ $x > c \quad f'(c) > 0$	

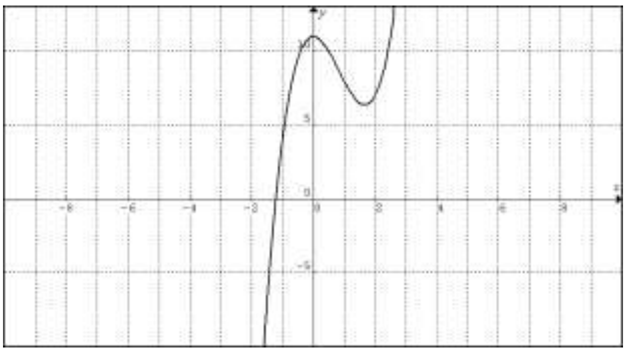
Another type of inflection point $x < c \quad f'(c) < 0$ $x = c \quad f'(c) = 0$ $x > c \quad f'(c) < 0$	

POINTS OF INFLEXION WILL HAVE A DERIVATIVE WITH A SQUARED FACTOR IN THEM

Example: $\frac{dy}{dx} = 3(x-2)^2$ thus x = 2 point of inflexion

If asked for the coordinates and the type of stationary points then you will need to do the following:

1. Find $f'(x) = 0$
2. Test either side of the stationary point for the values of the gradient
3. It probably is best to see how the graph looks and it is easy to see what type of stationary point it is.

Find all stationary points of the following graph $f(x) = 2x^3 - 5x + 11$	
Step 1- Always a good idea to graph it and see what is actually happening	
Step 2-	You can see immediately where the stationary points are: x=0 (max) and x=1.5 (min)

Step 3- Take the derivative of the graph And set the derivative to zero to find the stationary points	$f(x) = 2x^3 - 5x + 11$ $\rightarrow f'(x) = 6x^2 - 10x$ $\rightarrow$ Let $f'(x) = 0$ $\rightarrow 6x^2 - 10x = 0$ $\rightarrow 2x(3x-5) = 0$ $\rightarrow x = 0 \text{ or } x = \frac{5}{3}$
Now we need to show that for example x = 0 is a max point	
$x < 0 \quad f'(a) > 0$	Chose a value on the left of x=0 say x= -1 and put that into the equation for the derivative and see the value it gives you. $\rightarrow f'(x) = 6x^2 - 10x$ $\rightarrow = 6(-1)^2 - 10(-1)$ $\rightarrow = 6 + 10$ $\rightarrow = 16 > 0$
$x > 0 \quad f'(a) > 0$	$\rightarrow f'(x) = 6x^2 - 10x$ $\rightarrow = 6(1)^2 - 10(1)$ $\rightarrow = 6 - 10$ $\rightarrow = -4 < 0$
So do see that the gradient goes from a positive to a negative thus this means that x = 0 is a maximum. (We always consider from left to right)	X = 0 is a maximum point therefore as we can clearly see from the graph above
Repeat above procedure for other point to show it is a minimum	

If we try the same procedure for the other stationary point we will discover that it is a minimum.

Is there a quicker way of working out the nature of the stationary points? Graph them!

Skill Builder

	Answers
Find the stationary points of $f(x) = 4x^3 - 5x + 15$	
Find the stationary points of $f(x) = (3x-4)(x+3)(2x-2)$	

Concept – 15-Application of Calculus-Maximum and Minimum Problems

These problems can be quite challenging, so here below are the steps you will need to consider if you are to solve them successfully.

1. Draw diagram if relevant
2. Define variables
3. Determine variable to be maximized(or minimized) and set up equation for this variable
4. Calculate the derivative, set it to zero and solve
5. Determine the nature of the stationary point
6. Determine the value (like finding the y value)
7. Sketch the graph, watch the end point values

Example

Some psychologists believe that a numerical measure of a child's learning ability during the early years of life is approximately described by the function

$$L(x)=\frac{1}{0.6x\log_e x-x+2} \text{ for } 0<x\le 4\text{, where } x \text{ is the age in years.}$$

Questions	
Question: Use calculus to determine the age when a child's learning ability is best. Leave answer in exact form.	
Solution- Find the derivative of L(x) and set it to zero	$\frac{dL}{dx}=-\frac{0.6x\times\frac{1}{x}+0.6\log_e x-1}{(0.6x\log_e x-x+2)^2}$ $=-\frac{0.6\log_e x-0.4}{(0.6x\log_e x-x+2)^2}$ Let $\frac{dL}{dx}=0$, hence $0.6\log_e x-0.4=0$, $\log_e x=\frac{2}{3}$ or $x=e^{\frac{2}{3}}(=1.9477)$.
Question: . Determine the range of a child's learning ability between the age of 0 and 4. Leave answer in exact form	
Solution- Using the previous answers we need to sub into original equation and to determine L(x)	When $x=e^{\frac{2}{3}}$, max L $=\frac{1}{0.6e^{\frac{2}{3}}\times\frac{2}{3}-e^{\frac{2}{3}}+2}=\frac{1}{2-0.6e^{\frac{2}{3}}}$ $=\frac{5}{10-3e^{\frac{2}{3}}}(=1.2028)$. As $x\to 0$, $L\to\frac{1}{2}$. When $x=4$, $L=0.7535$. $\therefore$ the range is $\left(\frac{1}{2},\frac{5}{10-3e^{\frac{2}{3}}}\right]$.

A cubic function of the form $M(x)=a(x+b)(x-6.5)^2$ for $0<x\le 4$ can also be used to approximate a child's learning ability.	
Show that the maximum value of $M(x)$ is $4a\left(\frac{13+2b}{6}\right)^3$ when $x=\frac{13-4b}{6}$.	
	$\frac{dM}{dx}=2a(x+b)(x-6.5)+a(x-6.5)^2$ $=a(x-6.5)[2(x+b)+x-6.5]$ $=a(x-6.5)[3x+2b-6.5]$ for $0<x\le 4$. Let $\frac{dM}{dx}=0$, $3x+2b-6.5=0$, $x=\frac{13-4b}{6}$. Max M = $a\left(\frac{13-4b}{6}+b\right)\left(\frac{13-4b}{6}-6.5\right)^2$ $=a\left(\frac{13+2b}{6}\right)\left(\frac{2(13+2b)}{6}\right)^2$ $=4a\left(\frac{13+2b}{6}\right)^3$.
Find the values of a and b to 4 decimal places such that $L(x)$ and $M(x)$ have the same maximum value at the same age.	
	. Let $\frac{13+2b}{6}=e^{\frac{2}{3}}$, $b=0.3284$. Let $4a\left(\frac{13+2b}{6}\right)^3=\frac{5}{10-3e^{\frac{2}{3}}}$, $a=0.0255$.
Find the other two ages (2 decimal places) such that $L(x)$ and $M(x)$ give the same learning ability of a child in each case.	
	$M(x)=0.0255(x+0.3284)(x-6.5)^2$. Use graphics calculator to sketch both $L(x)$ and $M(x)$. Determine the x-coordinates of the two intersections, $x=0.59$ and $x=3.42$

Concept-16- Linear approximations

LINEAR APPROXIMATIONS

Using the following $f(x+h) \approx hf'(x) + f(x)$ we can find the approximation of certain function quickly, if x has been changed by a small amount *h*

$$f(x+h) \approx hf'(x) + f(x)$$

Value of function with original value

Approximate change in f(x) when x has been changed by h

New value

Example 2

If $f(x) = x^{\frac{1}{4}}$ use calculus to find approx value for $0.9^{\frac{1}{4}}$

Solution

Notice that the question is asking for the value of a function when the original x value changed from x=1 to x=0.9

X=1 h=-0.1 (small change in x)

$$f(x+h) \approx hf'(x) + f(x)$$

$$f(1-0.1) \approx -0.1 f'(1) + f(1)$$

$$f(1-0.1) \approx -0.1 \times \frac{1}{4} x^{\frac{-3}{4}} + f(1)$$

$$\rightarrow = \frac{-0.1}{4} + 1$$

$$\rightarrow = 0.975$$

Sometimes we are seeking an approximate change to y when x has been changed by a small amount δx then we need to use the following formula:

$$\delta y = \frac{dy}{dx} \times \delta x$$

Remember when answering questions involving linear approximations determine whether the question is asking for a change in y (δy , which equals $hf'(x)$ or $\frac{dy}{dx} \times \delta x$ or whether it is asking for a new value for the function $f(x+h)$

Example 3

The radius of a circle increases from 4cm to 4.01 cm. Find using calculus the approximate increase in area

Solution:

Observe that the question is asking for δA

$A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$ and $\delta r = 0.01$

$\rightarrow \delta A = \frac{dA}{dr} \times \delta r$

$\rightarrow \delta A = 2\pi(4) \times 0.01$

$\rightarrow \delta A = 0.08\pi$

Concept-17- Whether a function is differentiable at a point

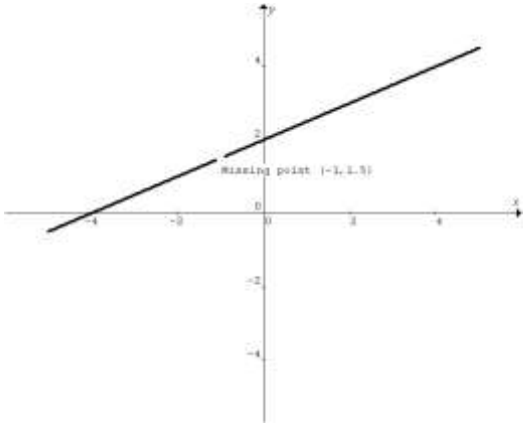
Limits and continuity

Sometimes the value of a function $f(x)$ at $x = a$ may not be defined, but $f(x)$ may get close to certain value L as x gets close to a from both sides of a. We say L is the limit of $f(x)$ as x approaches a. This idea is expressed by the notation, $\lim_{x \to a} f(x) = L$ or $\lim_{h \to 0} f(a+h) = L$, where $h = \Delta x$.

If x approaches a from the left, $x \to a^-$, h is a negative value.

If x approaches a from the right, $x \to a^+$, h is a positive value.

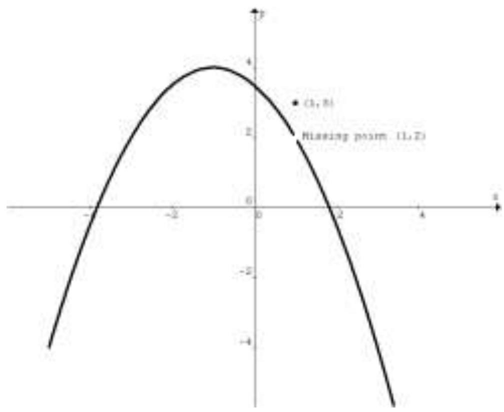
Example 1



The above function $f(x)$ is undefined at $x=-1$, i.e. $f(-1)$ does not exist. However, as $x \to -1$ from either side of $x=-1$, $f(x) \to 1.5$. $\therefore \lim_{x \to -1} f(x) = 1.5$ or $\lim_{h \to 0} f(-1+h) = 1.5$,
i.e. the limit of $f(x)$ exists as *x* approaches –1 and it is 1.5.

The function is **discontinuous** at $x=-1$.

Example 2

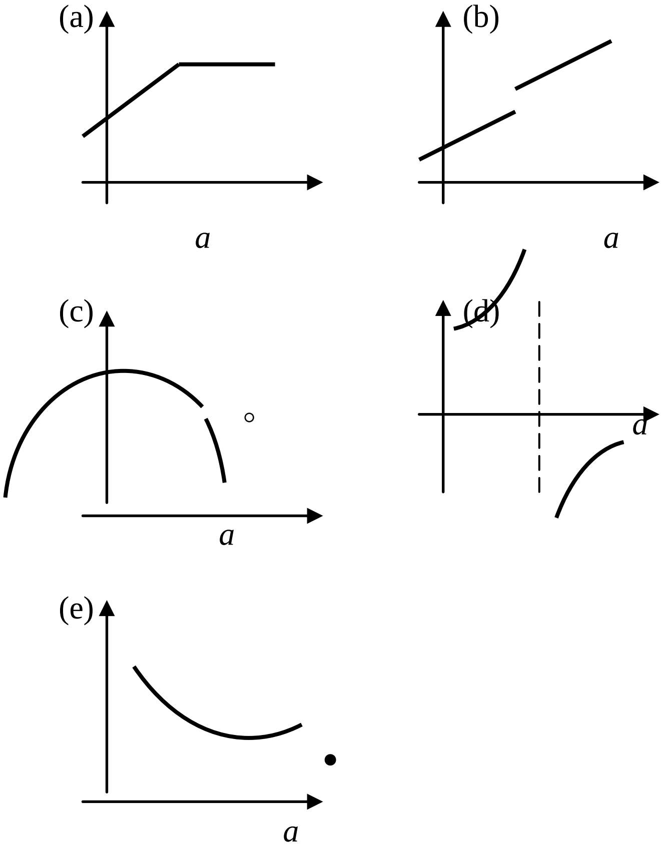


The function $f(x)$ is defined at $x=1$, and $f(1)=3$. However, as $x \to 1$ from either side of $x=1$, $f(x) \to 2$. $\therefore \lim_{x \to 1} f(x) = 2$ or $\lim_{h \to 0} f(1+h) = 2$, i.e. the limit of $f(x)$ exists as *x* approaches 1 and it is 2.
The function is discontinuous at $x=1$.

Differentiability of a function at a point on an interval

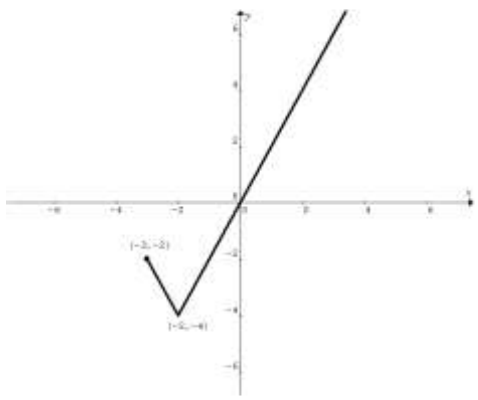
A function $f(x)$ is differentiable at $x=a$ if it is continuous and there is no abrupt change in its gradient at $x=a$, i.e. the section on the left of $x=a$ is *smoothly* joined to the section on the right, and the curve appears to be a straight section (**local linearity**) in the immediate neighbourhood of $x=a$.

A function is **not** differentiable at an end point. A function $f(x)$ is differentiable on a closed interval $[p,q]$ if $f(x)$ is differentiable at each point of the open interval (p,q).
Example 1 Discuss the differentiability of the following functions at $x=a$



(a) The function has an abrupt change in its gradient at $x = a$, $\therefore$ it is not differentiable at $x = a$.
(b) The function is not continuous at $x = a$, $\therefore$ it is not differentiable at $x = a$.
(c) The function is not continuous at $x = a$, $\therefore$ it is not differentiable at $x = a$.
(d) The function is undefined at $x = a$, $\therefore$ it is not differentiable at $x = a$.
(e) It is an end point of the function at $x = a$, $\therefore$ it is not differentiable at $x = a$.

Example 2 Discuss the differentiability of the following absolute function $f(x)$ on the unbounded interval $[-3, \infty)$.



$f(x)$ is differentiable at every point except the end point and the vertex. $\therefore$ $f(x)$ is differentiable on the interval $(-3, -2)$ or $(-2, \infty)$.

THESE NOTES DO NOT COVER ALL THE CONCEPTS STUDENTS WILL NEED TO KNOW IN CALCULUS- STAY TUNE FOR THE INTEGRATION SECTION