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Exponentials & Logarithmic Functions

V1

A brief overview of the topic for Year 12 Mathematical Methods

Exponential and Logarithmic Functions

In this topic we will learn the following concepts

1. How to plot and sketch exponential and logarithmic graphs
2. How to use the index laws to solve exponential and logarithmic functions
3. How to factorize these types of functions
4. Finding the inverses of exponentials and logarithmic functions
5. Applying theory to real life questions

Part 1: Sketching exponential and logarithmic functions

What are exponential functions?

Exponential functions are functions that look like this $f(x) = a^x$ where a $\in$ all positive real numbers except for 1.

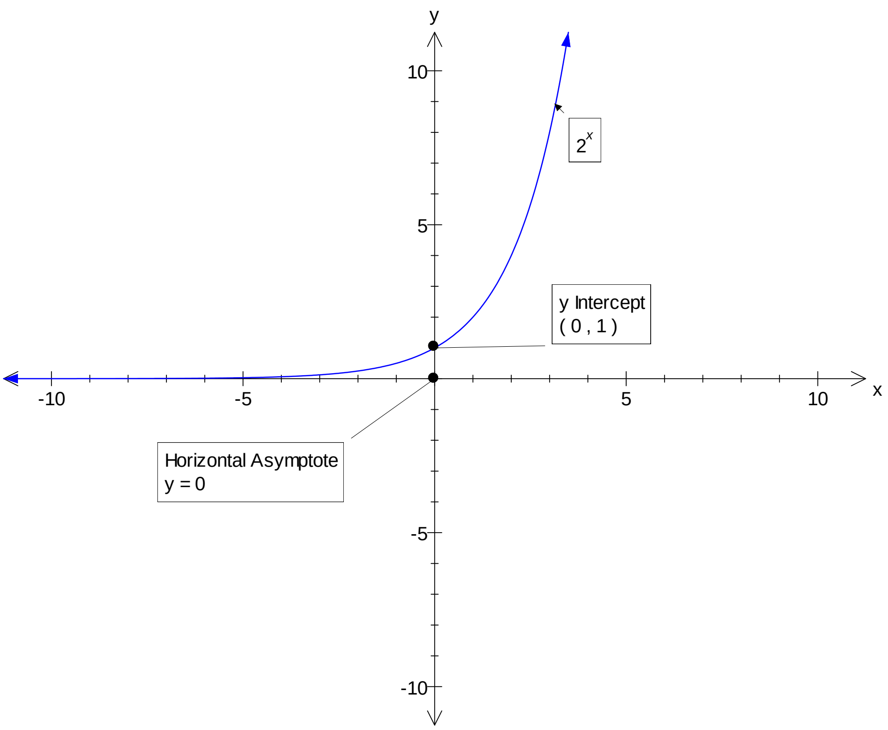
Here we are going to look at three cases of a

a =2, a = 10 and a = e

Let's briefly look at various graphs to help us visualize

Case 1 a = 2

Sketching the graph $y = 2^x$



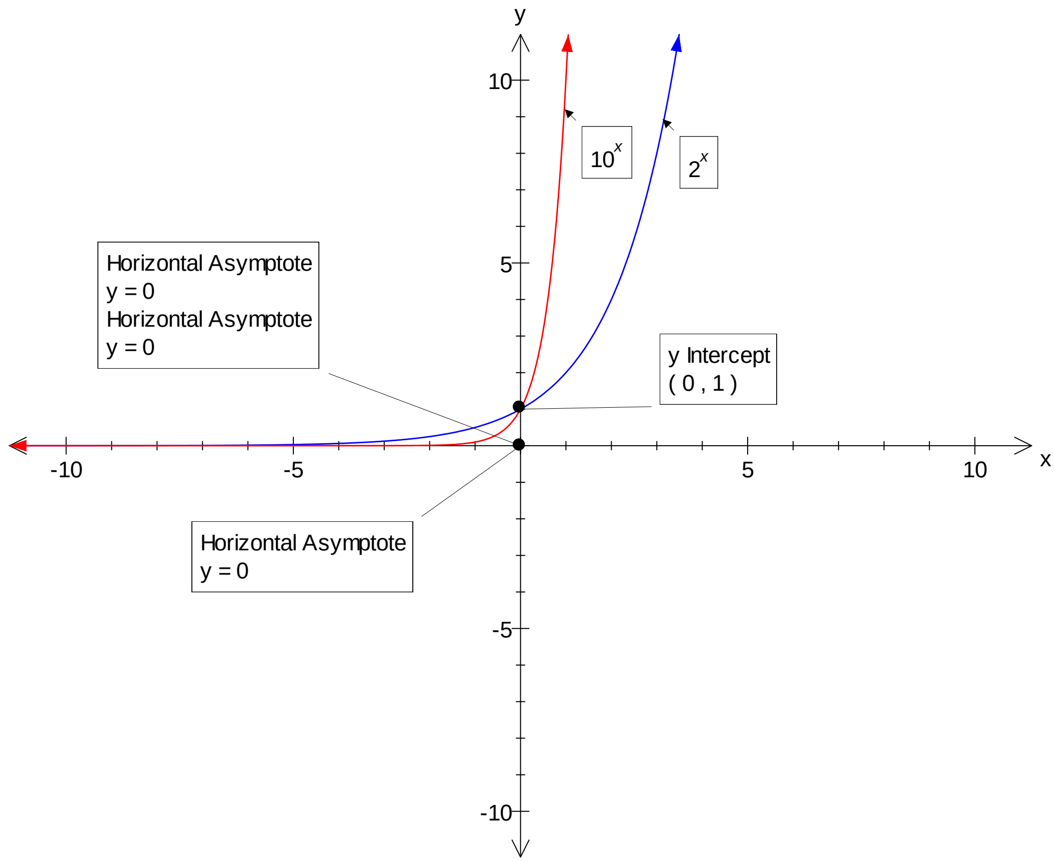
Notice from the above that this graph has the following properties:

Domain: R	Range: R+	Asymptote: y = 0	

ALL EXPONENTIAL FUNCTIONS ARE ONE TO ONE FUNCTIONS; THEREFORE THEY ALL HAVE AN INVERSE

Case 2 a = 10

Sketching the graph $y = 10^x$



Notice once again that the graph of $y = 10^x$ goes through the same y intercept and it has the same asymptote namely y = 0

So in general we can conclude the following:
Properties of this graph

$f(0) = 1$ if $x = 0$

If $x = 1$ then $f(1) = a$

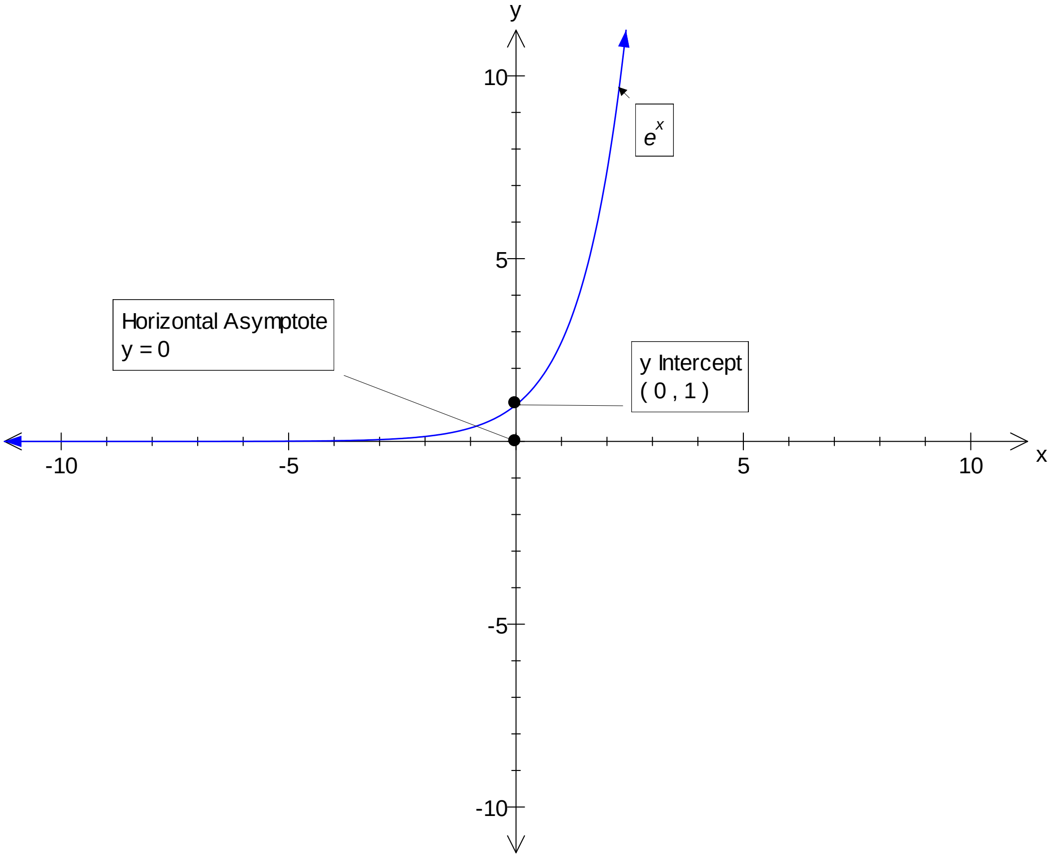
In general graphs of exponentials have the above form.

Case 3 a = e (2.71828)

This is a special number along other special numbers in mathematics such as 0 , 1, π and i. The number e is sometimes called Euler's number after the Swiss mathematician Leonhard Euler. What is special about e is that the value of the derivative (slope of the tangent line) of the function $y = e^x$ at the point x = 0 is exactly 1.

The function $y = e^x$ looks like the following below;

$y = e^x$



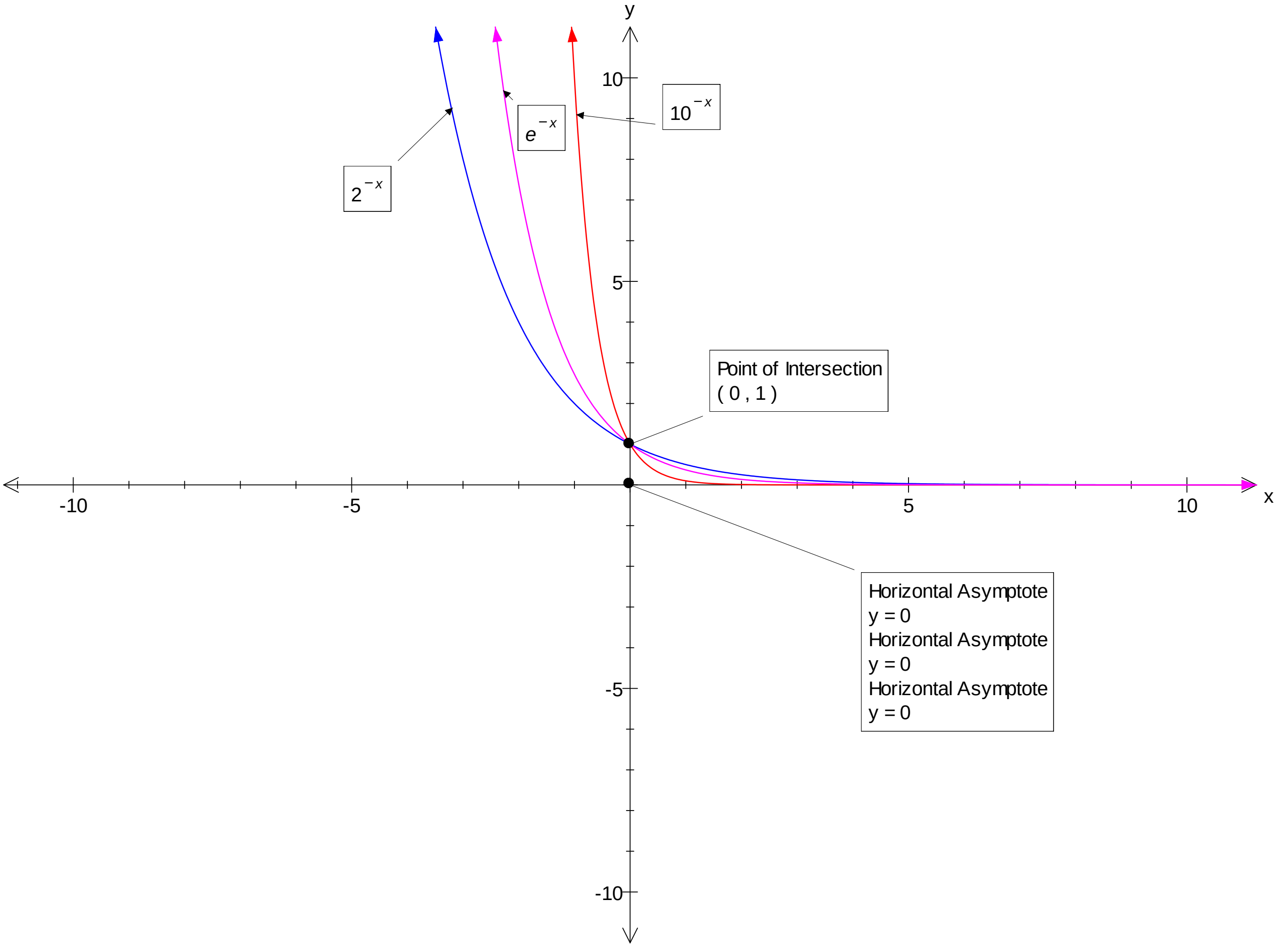
Once again we see the following characteristics for this graph

It passes through (0, 1) and there is an asymptote at y = 0

Notice the shape of the curve, it is upwards and it increases rapidly!

What about negative powers, how do the graphs look like then?

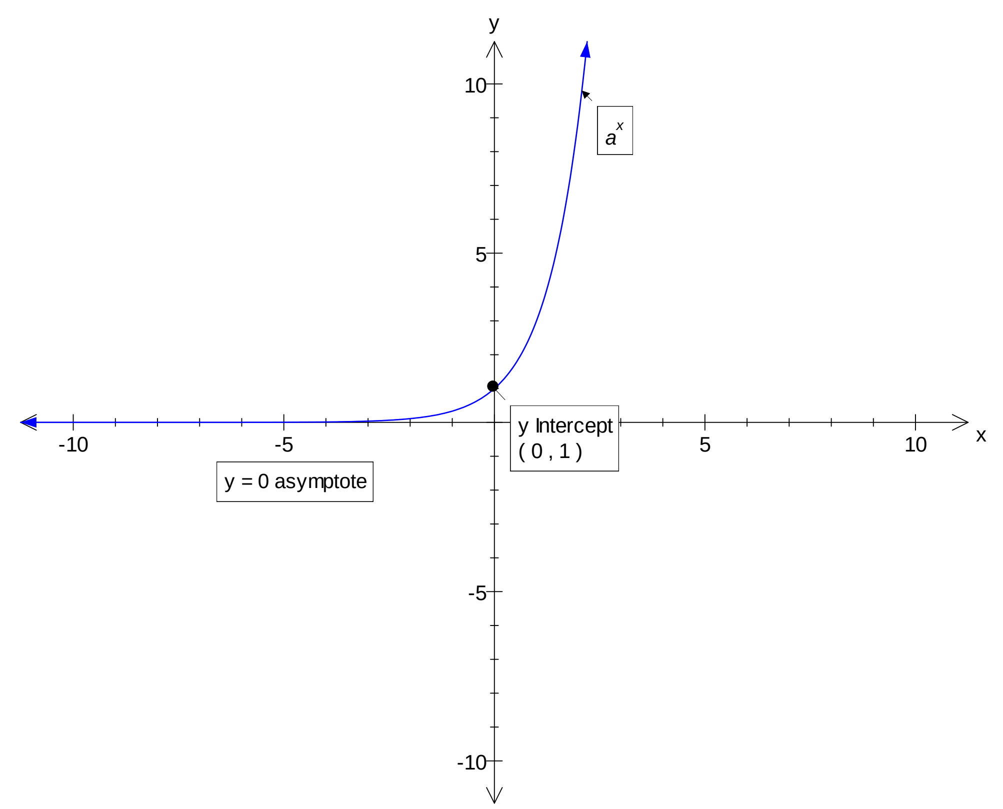
$y = 10^{-x}$ and $y = 2^{-x}$ and $y = e^{-x}$



What has happened? A reflection in the Y axis has occurred.

So we can make a few generalizations then

A graph of $y = a^x$ where a is a positive number will then look like the graph below



$f(0) = 1$ if $x = 0$

If $x = 1$ then $f(1) = a$

Asymptote y =0

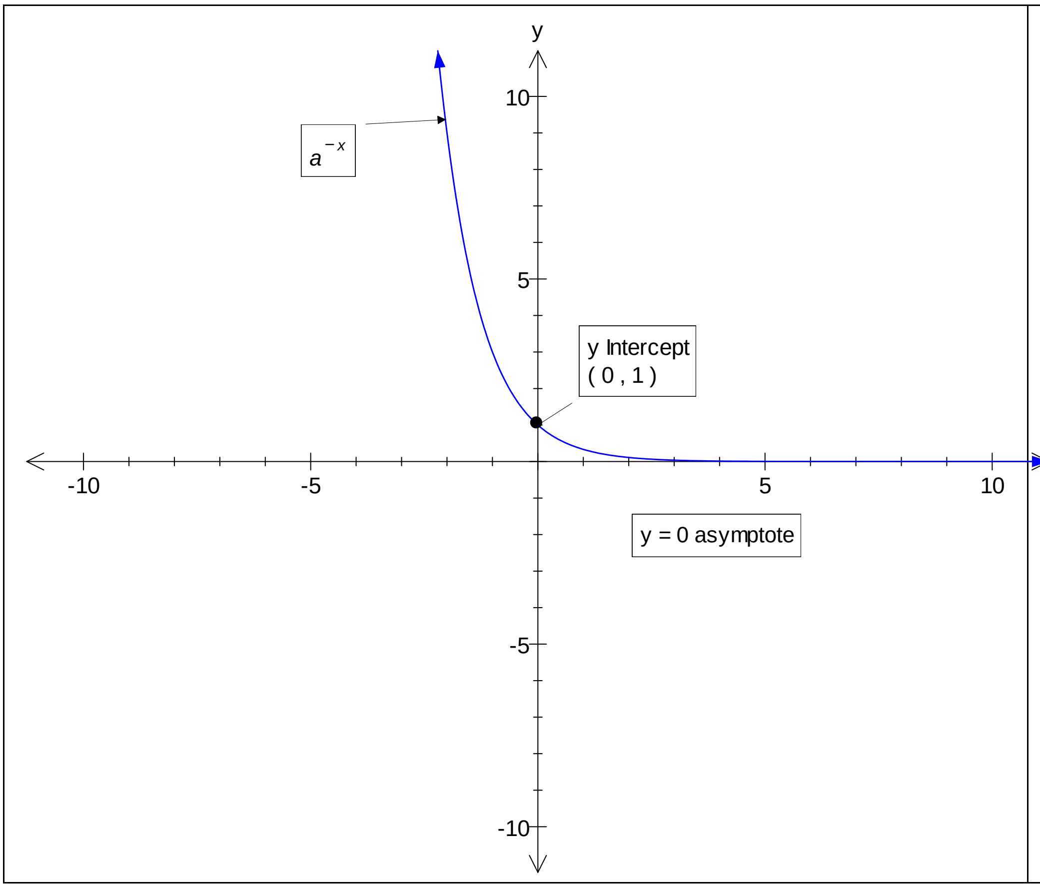
Y intercept= (0, 1)

Point: (1, a)

Domain: R

Range: R+

A graph of $y = a^{-x}$ where a is a positive number will then look like the graph below



$f(0) = 1$ if $x = 0$

If $x = 1$ then $f(1) = a$

Asymptote y =0

Y intercept: (0, 1)

Point: (-1, a)

Domain

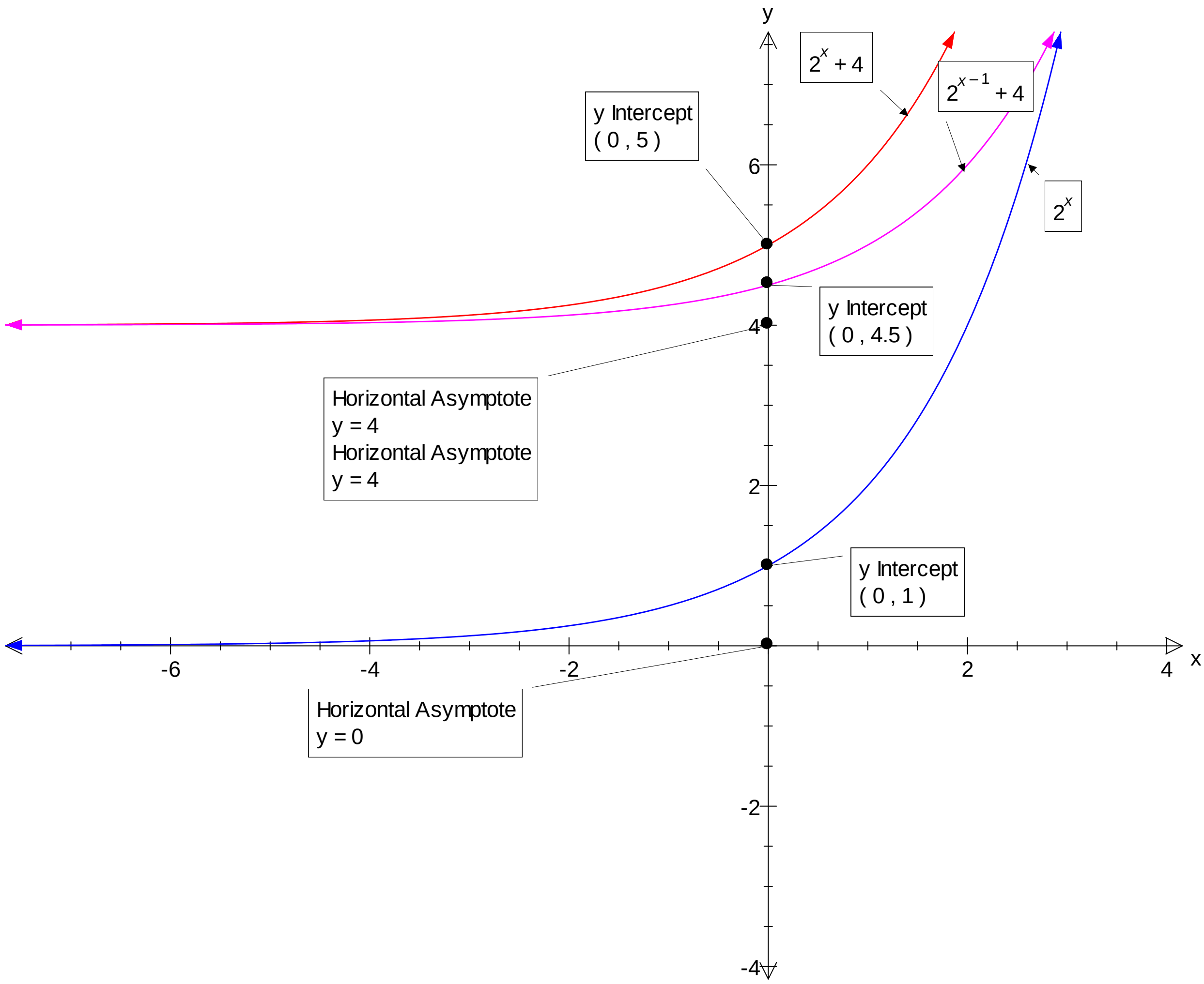
Range: R+

Let us investigate on our calculators what actually happens to the graph if there are a few translations occurring

1	$f(x) = 2^x$
2	$f(x) = 2^x \ \ +4$
3	$f(x) = 2^{(x-1)} \ \ +4$
4	$f(x) = ka^{(x-b)} + c\, ,\mathrm{k} > 0$

What would happen to the above graph if we had the following changes?

Answer



What conclusions can we draw?

What would happen to the above graph if we had the following changes?

1	$f(x) = 2^{3x}$
2	$f(x) = 2^{3x} \ \ +4$
3	$f(x) = 2^{3(x-1)} \ \ +4$

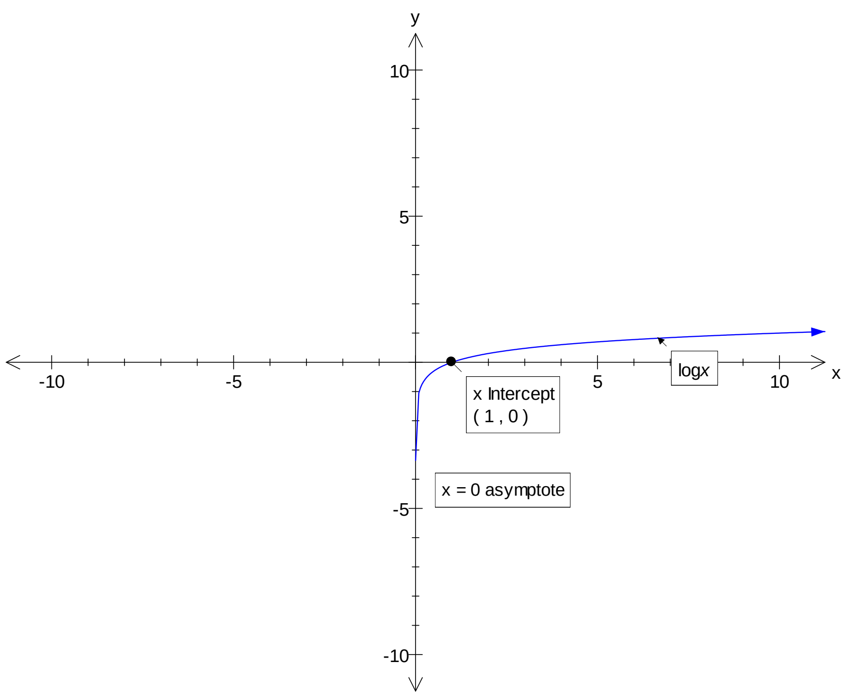
Remember that we mentioned that exponential functions are one to one functions and therefore they have an inverse

The inverse is the logarithmic function (we will explain a little about logarithms later, but for now we will examine how these graphs behave)

How to find the inverse?

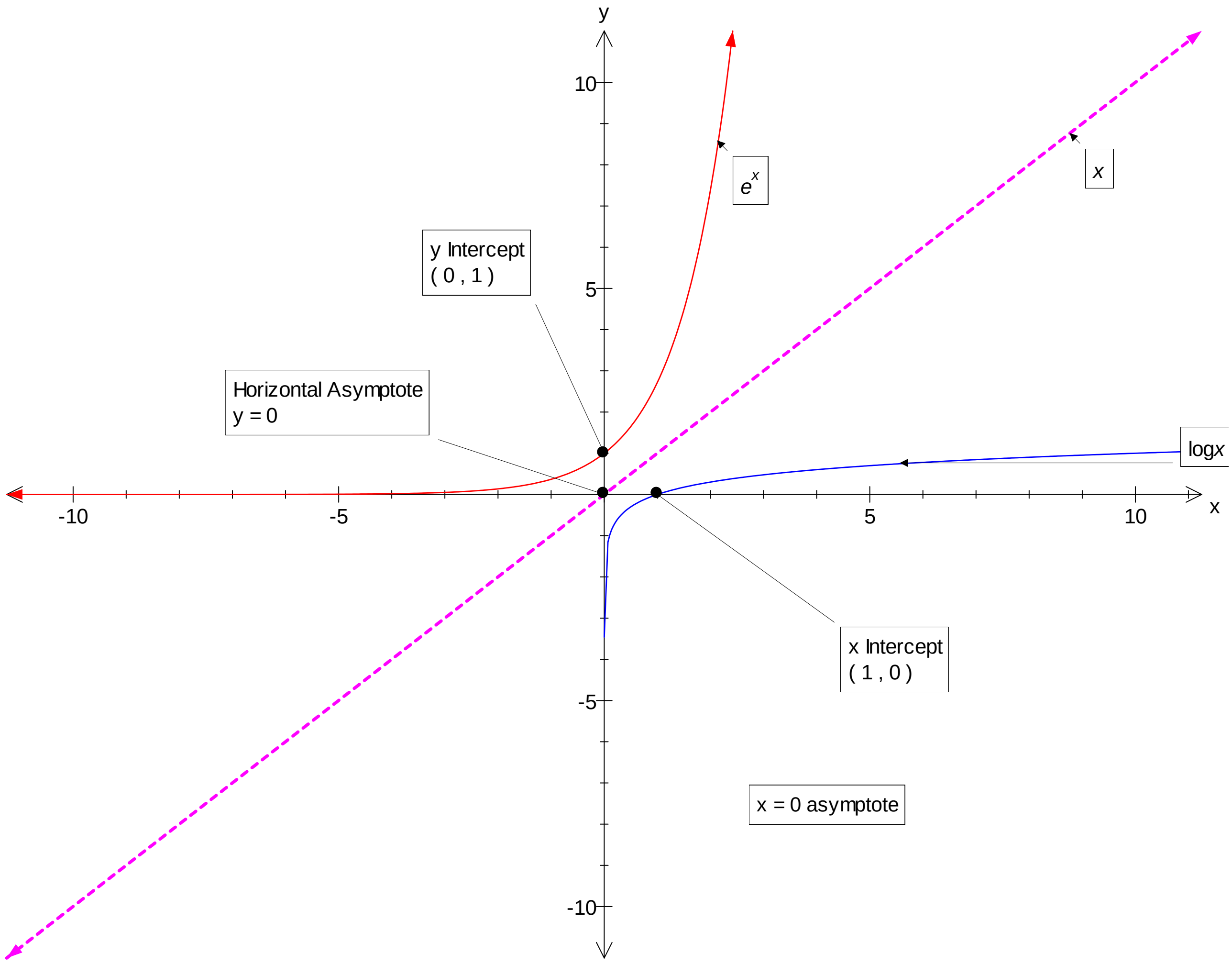
Steps	What happens
This is the function we start off with	$y = e^x$
If we draw it we will notice that it satisfies the rule of being a one to one function. A vertical line will cut it once A horizontal line will cut it once also Therefore it is a one to one function We can find the inverse	
To find the inverse we swap the x and y	$x = e^y$
Now how do we express the y as a function of x? This is where we take the logarithm of both sides	$\log_e x = \log_e e^y$ $\log_e x = y \log_e e$ $\log_e x = y$
So the inverse is then	$y = \log_e x$

Remember we will talk about index laws a little later on and then we will explain logarithms but for now we simply want to sketch them as a graph and see their characteristics
$y = \log x$



Now compare the graph of

$$y = e^x \text{ and the graph of } y = \log_e x$$



We can see that they have been reflected in the line $y = x$ (dashed line)
Notice how the asymptotes have been swapped and the x and y intercepts have swapped.

Let us now investigate some of the properties of the logarithmic graphs then

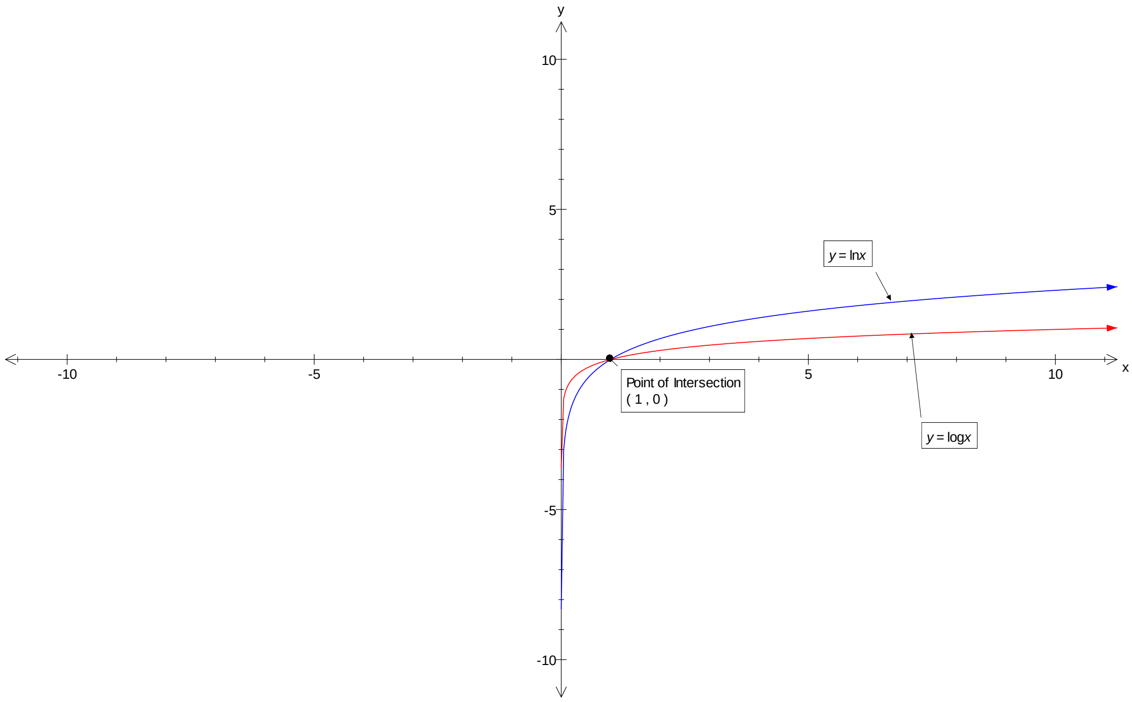
Quick Definitions

Natural logarithms	$\log_e$	*In*
Logs to the base of 10	$\log_{10}$	log

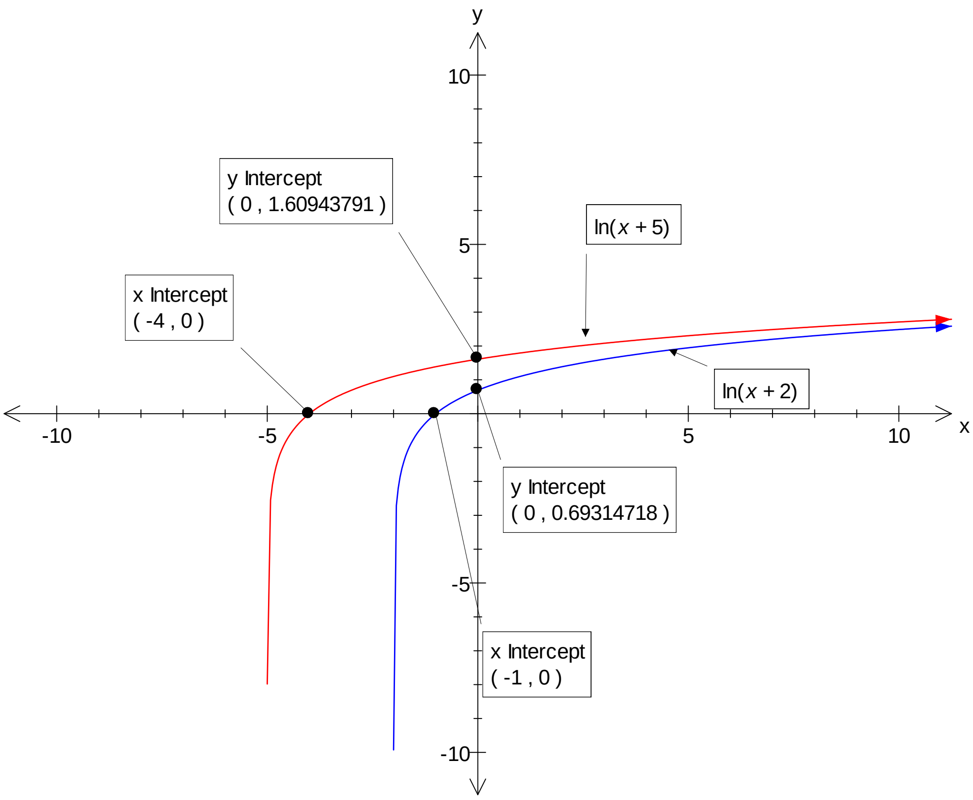
Sketch the following graphs using your graphics calculator

What happens or what do you observe?			
1	$y = \log_e x$ or $y = \ln x$	$y = \log_{10} x$	
2	$y = \log_e (x+2)$	$y = \log_e (x+5)$	
3	$y = \ln x$	$y = -\ln x$	
4	$y = -\log_{10} (x+2)$	$y = -\log_{10} (x+3)$	

Comparing graphs 1



Comparing graphs 2



How to use the index laws to solve exponential and logarithmic equations

Review of previous concepts

INDEX LAWS

Definitions	Explanations	Examples
a^m	**a** is called the base **m** is called the power or index	2^3, so the base is 2 and the power is 3 What this means is $2\times2\times2=8$
$a^m \times a^n = a^{m+n}$	Multiplying two together with the same base	$3^1 \times 3^{12} = 3^{1+12} = 3^{13}$
$\frac{a^m}{a^n} = a^{m-n}$	Dividing two with the same base	$\frac{4^3}{4^6} = 4^{3-6} = 4^{-3}$
$(a^m)^n = a^{mn}$	Raising an index to the power	$(5^2)^7 = 5^{14}$
$\frac{1}{a^m} = a^{-m}$	Finding the reciprocal	$\frac{1}{3^5} = 3^{-5}$
$a^0 = 1$	Definition – any base to power of 0 is always 1	$12^0 = 1$
$a^1 = a$	Definition	$15^1 = 15$

Try these examples to see how your skills are: Simplify them

	Simplify	Answer
1	$\frac{3x^6y^4}{12xy^3}$	$\frac{x^5y}{4}$
2	$\frac{(a^3)^4}{(a^2)^5}$	a^2
3	$x^2y^3 \times x^2y^4$	x^4y^7
4	$(5a^3b^4)^0$	1
5	$(b^4)^4 \div (b^2)^3$	b^{10}
6	$(3a)^0 + 3$	4
7	$3a^0 + 3$	6
8	$\frac{(ab)^{2n} - b^{3n}}{(ab^4)^n + b^{4n}}$	$= \frac{a^n - b^n}{b^{2n}}$

LOG LAWS

Definitions	Explanations	Examples
$\log_a p + \log_a q = \log_a(pq)$	When adding logs we multiply	$\log 3 + \log 4 = \log(3 \times 4) = \log 12$
$\log_a p - \log_a q = \log_a\left(\frac{p}{q}\right)$	When subtracting logs we divide first by second	$\log 12 - \log 3 = \log\left(\frac{12}{4}\right) = \log 3$
$\log_a p^n = n\log_a p$	Logs to the power	$\log 2^3 = 3\log 2$
$-\log_a q = \log_a\left(\frac{1}{q}\right)$	Negative logs	$\log\left(\frac{1}{4}\right) = -\log 4$
$\log_a 1 = 0$	Definition	
$\log_a a = 1$	Definition	
$\log_b p = \frac{\log_a p}{\log_a b}$	Interesting fact	
$\log_a x = \frac{\log_b x}{\log_b a}$ $a^x = b^{(\log_b a)x}$	Another way of expressing it	
$\log_a p$ is undefined for $p \le 0$	Definition from graphs	
$\log_a p < 0$ for $0 < p < 1$; $\log_a p > 0$ for $p > 1$	Definition from graphs	

Some explanations using examples will help us get our heads around logarithms

When we say the following

$\log_2 8 = 3$ this means $2^3 = 8$

Important
The 2 becomes the base on the other side of the equation
The 8 is the answer
The 3 is the power
So $\log_2 8 = 3$ this means $2^3 = 8$

Now let us try solving various logarithmic equations step by step

Example 1: Evaluate $5\log_2\left(\frac{1}{32}\right)$.

$$5\log_2\left(\frac{1}{32}\right)=5\log_2\left(\frac{1}{2^5}\right)=5\log_2 2^{-5}=-5\times 5\log_2 2=-25.$$

Example 2: Simplify $2\log_{10}(3x^2y)-3\log_{10}(2xy^2)$.

$$2\log_{10}(3x^2y)-3\log_{10}(2xy^2)=\log_{10}(3x^2y)^2-\log_{10}(2xy^2)^3$$
$$=\log_{10}(9x^4y^2)-\log_{10}(8x^3y^6)=\log_{10}\left(\frac{9x^4y^2}{8x^3y^6}\right)=\log_{10}\left(\frac{9x}{8y^4}\right).$$

Example 3: Evaluate $\log_3 10$.

Your calculator has only log (i.e. $\log_{10}$) and ln (i.e. $\log_e$).

$\log_3 10=\frac{\log_{10}10}{\log_{10}3}=2.0959$, or $\log_3 10=\frac{\log_e 10}{\log_e 3}=2.0959$.

Example 4: Show that $3\log_4 x-2\log_8 x=\frac{5}{6}\log_2 x$.

Change both logarithms on the left side of the identity to base 2.

$$3\log_4 x-2\log_8 x=\frac{3\log_2 x}{\log_2 4}-\frac{2\log_2 x}{\log_2 8}$$
$$=\frac{3\log_2 x}{\log_2 2^2}-\frac{2\log_2 x}{\log_2 2^3}=\frac{3}{2}\log_2 x-\frac{2}{3}\log_2 x$$
$$=\left(\frac{3}{2}-\frac{2}{3}\right)\log_2 x=\frac{5}{6}\log_2 x.$$

Example 5: Show that $\frac{1}{\log_e 10}=\log_{10} e$.

Change both sides to a common base *b*.

$$LHS=\frac{1}{\frac{\log_b 10}{\log_b e}}=\frac{\log_b e}{\log_b 10}.$$

$$RHS=\frac{\log_b e}{\log_b 10}.\ \therefore LHS=RHS.$$

Example 6: Given $10^p = e^q$, (i) find *q* in terms of *p*, (ii) find *p* in terms of *q*.

(i) $10^p = e^q$, $\log_e 10^p = \log_e e^q$, $p\log_e 10 = q\log_e e$, $\therefore q = p\log_e 10$.

(ii) Since $q = p\log_e 10$, $\therefore p = \dfrac{q}{\log_e 10} = q\log_{10} e$.

These two results show the way to change the base of an exponential function.

$10^p = e^{p\log_e 10}$; $e^q = 10^{q\log_{10} e}$. In general, $a^x = b^{x\log_b a}$.

Example 7: Change 5^x to base 10 and base *e*.

$5^x = 10^{x\log_{10} 5} \approx 10^{0.6990x}$; $5^x = e^{x\log_e 5} \approx e^{1.6094x}$.

More examples solving equations involving logarithms

Example 8

Find $\log_2 x = 5$

Solution

$\log_2 x = 5$

$x = 2^5$ ← Used one of the index laws

$x = 3$

$x = 32$

Example 9

Solve $\log_2(2x-1) = 4$

Solution:

$\log_2(2x-1) = 4$

$\therefore (2x-1) = 2^4$

$\to 2x - 1 = 16$

$\to 2x = 17$

$\to \dfrac{2x}{2} = \dfrac{17}{2}$

$\to x = \dfrac{17}{2}$

Example 10

Solve $\log_e(3x+1)=0$

Solution:

$$\log_e(3x+1)=0$$
$$\rightarrow 3x+1=e^0$$
$$\rightarrow 3x+1=1$$
$$\rightarrow 3x=0$$
$$\rightarrow x=0$$

Example 11

Solve $\log_e(x-1)+\log_e(x+2)=\log_e(6x-8)$

Solution:

$$\log_e(x-1)+\log_e(x+2)=\log_e(6x-8)$$
$$\rightarrow \log_e(x-1)(x+2)=\log_e(6x-8)$$

Now you can see that we can equate the algebra part since the base is the same

$$(x-1)(x+2)=(6x-8)$$
$$x^2+x-2=6x-8$$
$$x^2-5x+6=0$$
$$(x-3)(x-2)=0$$
$$x-3=0 \quad \text{and} \quad x-2=0$$
$$x=3 \quad \text{or} \quad x=2$$

Example 12

Solve $4\log_e(2x-5)=7$

Solution:

$$4\log_e(2x-5)=7$$
$$\rightarrow \log_e(2x-5)=\frac{7}{4}$$
$$\rightarrow \log_e(2x-5)=1.75$$
$$\rightarrow (2x-5)=e^{1.75}$$
$$\rightarrow 2x-5=5.755$$
$$\rightarrow x=5.377$$

Example 13

Solve $5(0.27^x) = 102$

Solution:

This is important YOU MUST TAKE NOTICE OF HOW THIS IS DONE

$5(0.27^x) = 102$

$\rightarrow \frac{5(0.27^x)}{5} = \frac{102}{5}$

$\rightarrow 0.27^x = 20.4$

$\rightarrow$ Here we will take the log on both sides !!

$\rightarrow \log_e 0.27^x = \log_e 20.4$

$\rightarrow x\log_e 0.27 = \log_e 20.4$

$\rightarrow \frac{x\log_e 0.27}{\log_e 0.27} = \frac{\log_e 20.4}{\log_e 0.27}$

$\rightarrow x = -2.3$

IMPORTANT: WE COULD HAVE TAKEN $\log_e$ OR $\log_{10}$ IT DOES NOT REALLY MATTER.

Whenever you see log it means $\log_{10}$ and when you see ***"ln"*** it means $\log_e$

More examples

Example 14: Solve the following equations.

(i) $\log_2 x = 5$ (ii) $\log_2(x^2 - 1) = 3$ (iii) $\log_2\left(\frac{1}{x}\right) = 1.5$

(i) $\log_2 x = 5$, the equivalent is $x = 2^5$, $x = 32$.

(ii) $\log_2(x^2 - 1) = 3$, the equivalent is $x^2 - 1 = 2^3$, $x^2 = 9$, $x = \pm 3$.

(iii) $\log_2\left(\frac{1}{x}\right) = 1.5$, the equivalent is $\frac{1}{x} = 2^{1.5}$, $x = 2^{-1.5}$ or $2^{-\frac{3}{2}}$ or 0.3536

Example 15: Solve $\log_x \sqrt{243} = 2.5$.

$\log_x \sqrt{243} = 2.5$, the equivalent is $\sqrt{243} = x^{2.5}$, $\sqrt{3^5} = x^{2.5}$, $(3^5)^{\frac{1}{2}} = x^{2.5}$, $3^{2.5} = x^{2.5}$, $\therefore x = 3$

Example 16: Solve $\log_x 5x = 3$, where $x > 0$.

$\log_x 5x = 3$, $\log_x 5 + \log_x x = 3$, $\log_x 5 + 1 = 3$, $\log_x 5 = 2$, the equivalent is $x^2 = 5$, $\therefore x = \sqrt{5}$.

Example 17: Solve $\log_{10} x - \log_{10}(x+1) = 1$.

$\log_{10} x - \log_{10}(x+1) = 1$, $\log_{10}\left(\frac{x}{x+1}\right) = 1$, the equivalent is $\frac{x}{x+1} = 10^1$, $\therefore x = 10(x+1)$, $x = 10x + 10$, $9x = -10$, $x = -\frac{10}{9}$.

Example 18: Solve $\log_{10}\left(\sqrt{35} + x\right) + \log_{10}\left(\sqrt{35} - x\right) = 1$.

$\log_{10}\left(\sqrt{35} + x\right) + \log_{10}\left(\sqrt{35} - x\right) = 1$, $\log_{10}\left(\sqrt{35} + x\right)\left(\sqrt{35} - x\right) = 1$,
the equivalent is $\left(\sqrt{35} + x\right)\left(\sqrt{35} - x\right) = 10^1$,
$\therefore 35 - x^2 = 10$, $x^2 = 25$, $x = \pm 5$.

Example 19: Solve $2\log_e(x-2) - \log_e(x+1) = 0$.

$2\log_e(x-2) - \log_e(x+1) = 0$, $\log_e \frac{(x-2)^2}{x+1} = 0$,
$\frac{(x-2)^2}{x+1} = 1$, $(x-2)^2 = x+1$, $x^2 - 4x + 4 = x + 1$,
$x^2 - 5x + 3 = 0$, $\therefore x = \frac{5 \pm \sqrt{25-12}}{2}$,
i.e. $x = \frac{5+\sqrt{13}}{2}$ or $\frac{5-\sqrt{13}}{2}$. Only the first solution is correct because $x > 2$ for $2\log_e(x-2)$ to be defined.

Example 20: $(-2,1)$ is a point on the curve $y = 2\log_e(1-ax)$. Find a.

$(-2,1) \to 1 = 2\log_e(1 - a(-2))$, $\frac{1}{2} = \log_e(1+2a)$,
$1 + 2a = e^{\frac{1}{2}}$, $a = \frac{1}{2}\left(\sqrt{e} - 1\right)$.

Example 21: Find a and b such that $y = \log_e \sqrt{ax+b} + 1$ passes through the points $(0,1)$ and $\left(1, \frac{3}{2}\right)$.

$(0,1) \to 1 = \log_e \sqrt{b} + 1$, $\therefore \log_e \sqrt{b} = 0$, $\sqrt{b} = 1$, $\therefore b = 1$ ……(1)
$\left(1, \frac{3}{2}\right) \to \frac{3}{2} = \log_e \sqrt{a+b} + 1$, $\log_e \sqrt{a+b} = \frac{1}{2}$, $\sqrt{a+b} = e^{\frac{1}{2}}$, $\therefore a + b = e$ ……(2)
Substitute eq.(1) in eq.(2), $a = e - 1$.

Problems involving exponentials

Example 22: Solve the following equations.
(a) $4^x = 8$ (b) $10^{2x-3} = 0.01$ (c) $4^{x+1} = 10^{x-1}$.

(a) $4^x = 8$, $(2^2)^x = 2^3$, $2^{2x} = 2^3$, $\therefore 2x = 3$, $x = \frac{3}{2}$.

(b) $10^{2x-3} = 0.01$, $10^{2x-3} = \frac{1}{10^2}$, $10^{2x-3} = 10^{-2}$,

$\therefore 2x - 3 = -2$, $x = \frac{1}{2}$.

(c) $4^{x+1} = 10^{x-1}$, 4 and 10 cannot be changed to the same base. Take the log (to base 10 is simpler than to base e in this case) of both sides of the equation. $\log_{10} 4^{x+1} = \log_{10} 10^{x-1}$,
$(x+1)\log_{10} 4 = x - 1$, $(x+1)0.6021 = x - 1$, $0.6021x + 0.6021 = x - 1$, $x = 4.0259$.

Example 23: Solve $5e^{3x+2} = 6$.

$5e^{3x+2} = 6$, $e^{3x+2} = 1.2$, the equivalent is $3x + 2 = \log_e 1.2$,
$3x = \log_e 1.2 - 2$, $\therefore x = \frac{1}{3}(\log_e 1.2 - 2)$ in exact form or $x \approx -0.6059$.

Example 24: Solve $8e^{x+2} = 3e^{2-x}$.

$8e^{x+2} = 3e^{2-x}$, $\frac{e^{x=2}}{e^{2-x}} = \frac{3}{8}$, $e^{x-2-(2-x)} = \frac{3}{8}$, $e^{2x} = \frac{3}{8}$, the equivalent is $2x = \log_e\left(\frac{3}{8}\right)$, $x = \frac{1}{2}\log_e\left(\frac{3}{8}\right)$ in exact form or
$x \approx -0.4904$.

Alternative method: $8e^{x+2} = 3e^{2-x}$, $8e^{x+2} - 3e^{2-x} = 0$, $e^{2-x}(8e^{2x} - 3) = 0$. Since $e^{2-x} \neq 0$, $\therefore 8e^{2x} - 3 = 0$, $e^{2x} = \frac{3}{8}$ etc.

Example 25: Solve $3e^{2x} - 4e^x + 1 = 0$.

$3(e^x)^2 - 4(e^x) + 1 = 0$,
$(3e^x - 1)(e^x - 1) = 0$, [factorise by trial and error]
either $3e^x - 1 = 0$ or $e^x - 1 = 0$,
$\therefore e^x = \frac{1}{3}$ or $e^x = 1$, i.e. $x = \log_e\left(\frac{1}{3}\right)$ or $x = 0$.

Example 26: Solve $3e^{2x} + 4e^x + 1 = 0$.

This equation has no real solutions for x, because all three terms are greater than 0, $\therefore$ sum > 0.

Example 27: Solve $e^{2x}-4e^{x}-4=0$.

$(e^{x})^{2}-4(e^{x})-4=0$ cannot be factorised over Q, set of rational numbers, $\therefore$ use the quadratic formula to obtain $e^{x}=\dfrac{-(-4)\pm\sqrt{(-4)^{2}-4(1)(-4)}}{2(1)}$, $e^{x}=2\pm2\sqrt{2}$. Since $e^{x}>0$, $\therefore e^{x}=2+2\sqrt{2}$, hence $x=\log_{e}(2+2\sqrt{2})$ in exact form or $x\approx1.5745$.

Example 28: Find k such that the curve $y=3e^{-kx+2}$ passes through $(-2,3)$.

When $x=-2$, $y=3$, $\therefore 3=3e^{2k+2}$, $e^{2k+2}=1$,
$2k+2=0$, $\therefore k=-1$.

Example 29: Find a and b such that the curve $y=ae^{bx}+1$ passes through the points $(-1,2)$ and $(1,4)$.

Use the two points to set up two simultaneous equations:
$(-1,2)\rightarrow ae^{-b}+1=2$, $\therefore ae^{-b}=1$ ………(1)
$(1,4)\rightarrow ae^{b}+1=4$, $\therefore ae^{b}=3$ ………….(2)
Eq.(2)/eq.(1), $\dfrac{ae^{b}}{ae^{-b}}=\dfrac{3}{1}$, $e^{2b}=3$, $2b=\log_{e}3$, $\therefore b=\dfrac{1}{2}\log_{e}3=\log_{e}\sqrt{3}$ ……(3)
Substitute eq.(3) in eq.(2), $ae^{\log_{e}\sqrt{3}}=3$, $a\sqrt{3}=3$, $\therefore a=\dfrac{3}{\sqrt{3}}=\sqrt{3}$.

Example 30: Find b and c such that the curve $y=2e^{bx}+c$ passes through the points $(2,6)$ and $(4,10)$.

$(2,6)\rightarrow 2e^{2b}+c=6$, $\therefore 2e^{2b}=6-c$ ………...(1)
$(4,10)\rightarrow 2e^{4b}+c=10$, $\therefore 2e^{4b}=10-c$ ……..(2)
Eq.(2)/eq.(1), $\dfrac{2e^{4b}}{2e^{2b}}=\dfrac{10-c}{6-c}$, $\therefore e^{2b}=\dfrac{10-c}{6-c}$ ………(3)
Substitute eq.(3) in eq.(1), $2\left(\dfrac{10-c}{6-c}\right)=6-c$, $\therefore 2(10-c)=(6-c)^{2}$, $(6-c)^{2}-2(10-c)=0$, $(6-c)^{2}-2(6-c)-8=0$.
Factorise to obtain $[(6-c)-4][(6-c)+2]=0$.
Hence either $6-c-4=0$, i.e. $c=2$, or $6-c+2=0$, i.e. $c=8$. The second result is not possible because it leads to an impossibility $2e^{2b}=-2$. $\therefore c=2$ ……(4)
Substitute eq.(4) in eq.(1), $2e^{2b}=4$, $e^{2b}=2$, $2b=\log_{e}2$,
$b=\dfrac{1}{2}\log_{e}2=\log_{e}\sqrt{2}$.

Final Note
We cannot cover every situation especially when it comes to solving problems involving exponentials or logarithms.
The important thing to remember is to look for ways to express them as quadratics or to express them into simpler expressions using the index laws or the log laws.