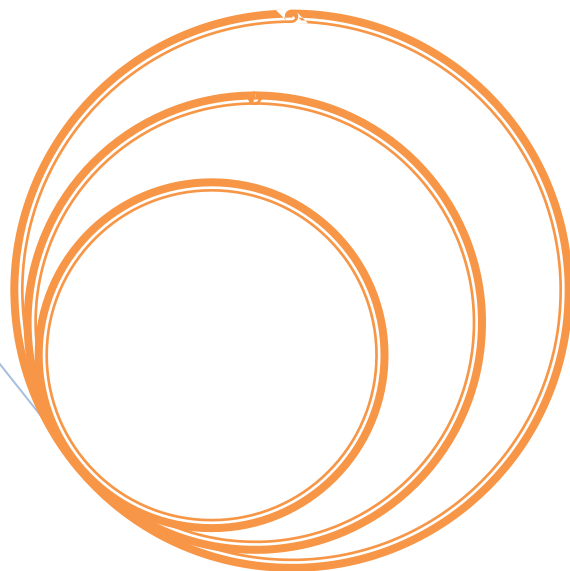


Updated- 10 May 2011

All reasonable efforts have been made to make sure the notes are accurate. The author cannot be held responsible for any damages arising from the use of these notes in any fashion

Calculus Part 2-v2

Antidifferentiating and Integration



Calculus-Calculus-Antidifferentiating and Integration

Calculus is a huge topic. In our previous introduction we saw how we can take the derivatives of various functions. Now we will go backwards, meaning given a differentiated function we go back to find the original function. This is called antidifferentiation.

So what we will consider are the following:

1. Antidifferentiating is the inverse operation of differentiation
2. Sometimes we use the expression 'integration' which basically means the same as Antidifferentiation, except we tend to use that expression when we are actually evaluating expressions.
3. Applications:

To find the area under a graph and the x axis

To find the area bounded by two or more curves

Concept-1-Approximations to finding the area under a curve

In the same way that we used first principles to approximately find the gradient of the tangent to a curve we can also use approximations in doing integrations.

There are 3 standard methods of approximations:

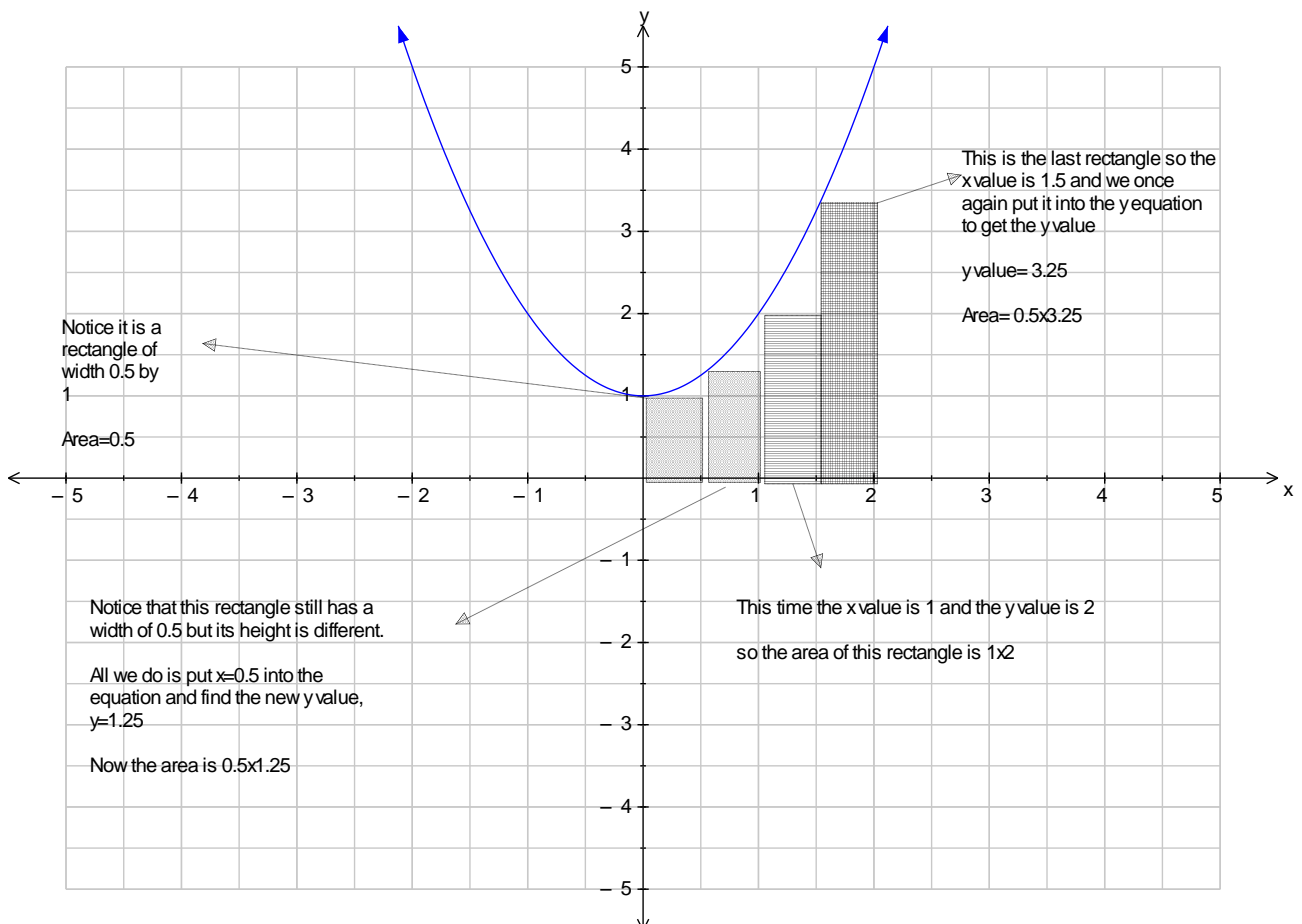
1. The left hand endpoint method
2. The right hand endpoint method
3. The trapezoidal method

Concept 1-a-The left hand endpoint method

Take the graph with equation $y = x^2 + 1$

We can approximate the area under the curve using rectangles. Here our rectangles will have a width of 0.5 units.

To use the left hand endpoint, rectangles are drawn so that the left hand top corner of the rectangle is in contact with the curve.



The area is approximated by calculating the area of the rectangles formed

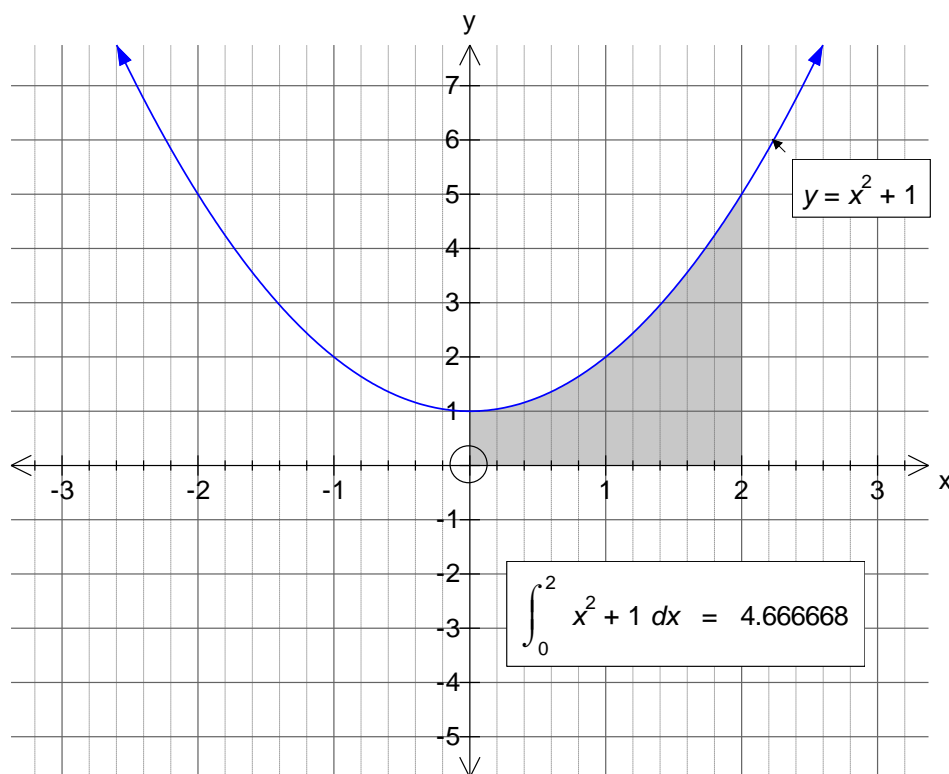
In this example:

$$A = (0.5 \times 1) + (0.5 \times 1.25) + (0.5 \times 2) + (0.5 \times 3.25)$$

$$A = 3.75 \text{ sq units}$$

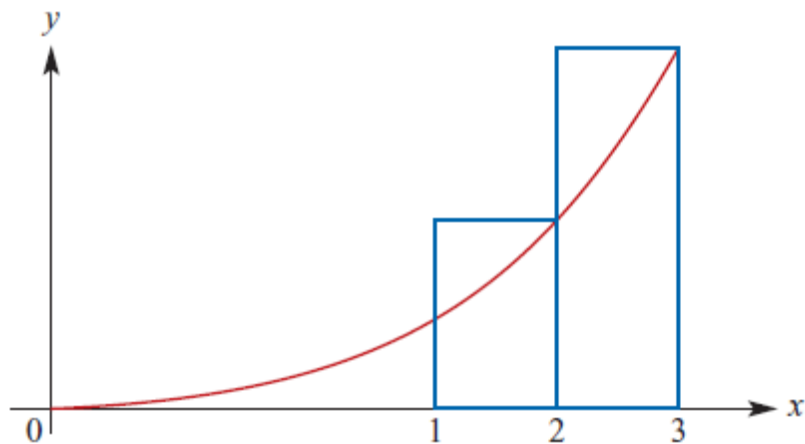
X	Y
0	1
0.5	1.25
1	2
1.5	3.25
2	5

Later you will learn to do the most accurate method of finding the area, but this is what you should really get:



Concept 1-b-right hand endpoint method

Same method as before except that the rectangles are drawn so that the right hand top corner of the rectangles is in contact with the curve

**Concept 1-c-the trapezoid method**

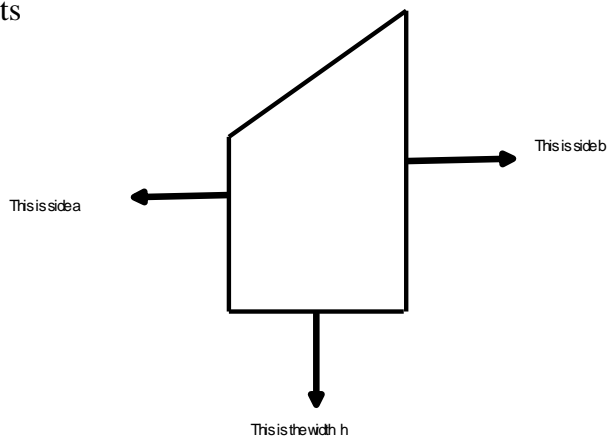
This method uses trapeziums to approximate the area and is the most accurate of the 3 methods.

Formula for a trapezium $A = \frac{(a+b)h}{2}$

So the area this time is the following:

$$A = \left(\frac{(1+1.25)0.5}{2}\right) + \left(\frac{(1.25+2)0.5}{2}\right) + \left(\frac{(2+3.25)0.5}{2}\right) + \left(\frac{(3.25+5)0.5}{2}\right)$$

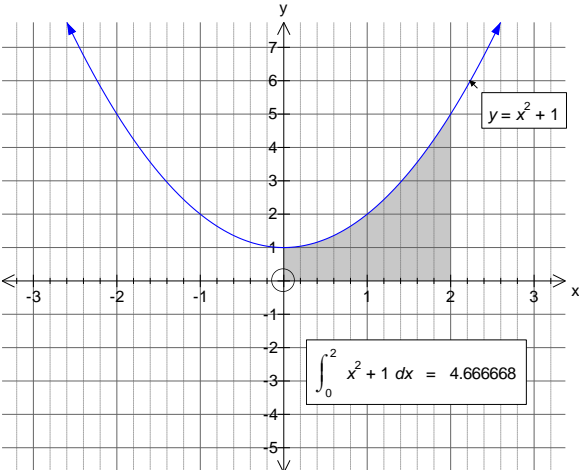
$$A = 4.75 \text{ sq units}$$



Obviously the trapezoid method is much more accurate than the other two methods, but how do we do it properly

Concept- Learning to take the area under a graph

Find the area between 0 and 2 for the graph $y = x^2 + 1$	Steps to take
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<p>Sketch the graph to see what we are looking out</p> <p>Notice we are looking to find the area between the graph and the x-axis between $x = 0$ and $x = 2$</p>	
<p>To show this in calculus we use the following symbol- $\int y = x^2 + 1$</p> <p>This is the integral symbol- \int</p> <p>Now we put the x values always starting from left to right on the graph</p>	$\int_0^2 (x^2 + 1) dx$ <p>So this basically means lets integrate this function and find the area between $x = 0$ and $x = 2$ with respect to x</p>
<p>Now when we integrate we normally do the opposite of what we did in differentiation and we will see all the rules for this later.</p>	$A = \int_0^2 (x^2 + 1) dx$ $\rightarrow = \left[\frac{x^3}{3} + x \right]_0^2$ $\rightarrow = \left(\frac{2^3}{3} + 2 \right) - \left(\frac{0^3}{3} + 0 \right)$ $\rightarrow = \frac{2^3}{3} + 2$ $\rightarrow = \frac{8}{3} + 2$ $\rightarrow = \frac{14}{3}$ $\rightarrow = 4 \frac{2}{3} \text{ sq units}$

INTEGRAL CALCULUS

So integral calculus is the process of antidifferentiating, and when it is used on functions it helps us calculate accurately the area under the curve.

Symbols

The antidifferentiation of a function $f(x)$ worked out between limits a and b is written as follows

$$\int_a^b f(x)dx = [F(X)]_a^b = F(b) - F(a)$$

Notice F(X) is the antiderivative of the original f(x)

Now before we go any further we can state in general terms that integrals can be of 3 types

Type 1- INDEFINITE INTEGRAL

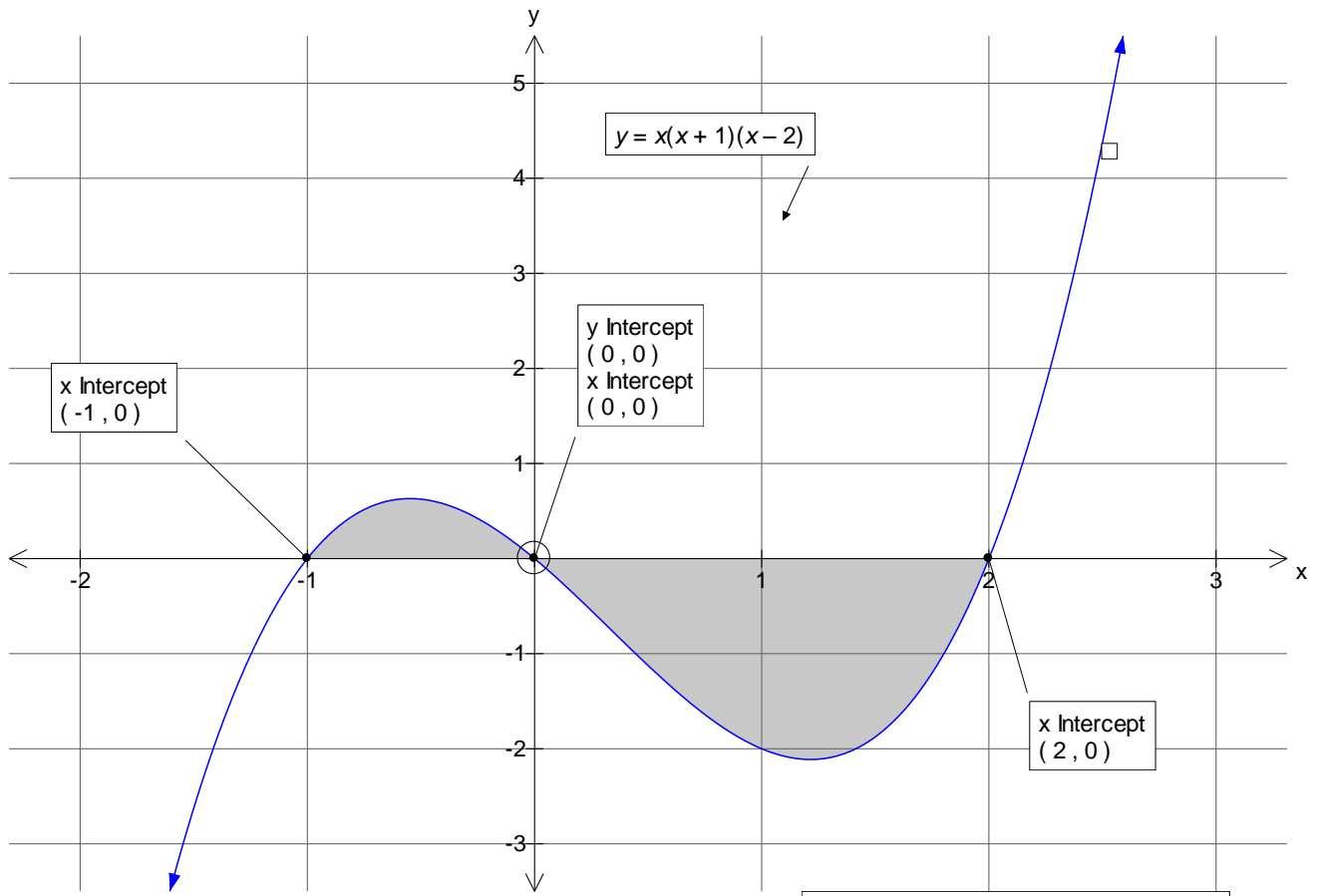
- These integrals have no limits involved
- So in the answer once we have found the antiderivative we add the term +C (stands for a constant).
- $\int x^3 dx = \frac{x^4}{4} + c$

Type 2-DEFINITE INTEGRAL

- These integrals have limits in them
- So there will be an exact answer (or a decimal answer)
- $\int_0^{\pi} (\sin x) dx$

Type 3-AREA UNDER A CURVE

- If we are asked to calculate the area bound by a curve and the x axis, we must integrate the function using the x axis intercepts as the limits
- Always draw the graph carefully
- When calculating areas we must draw the graph because sometimes the area we obtain will be negative since it is under the x axis and then a different part is above the x axis and the area of that section will be positive. If we are not careful the area can cancel each other out



Notice from the above example we must be very careful when we are working out area as the example above highlights.

So in the example above we must work out the area separately so that the areas do not cancel out

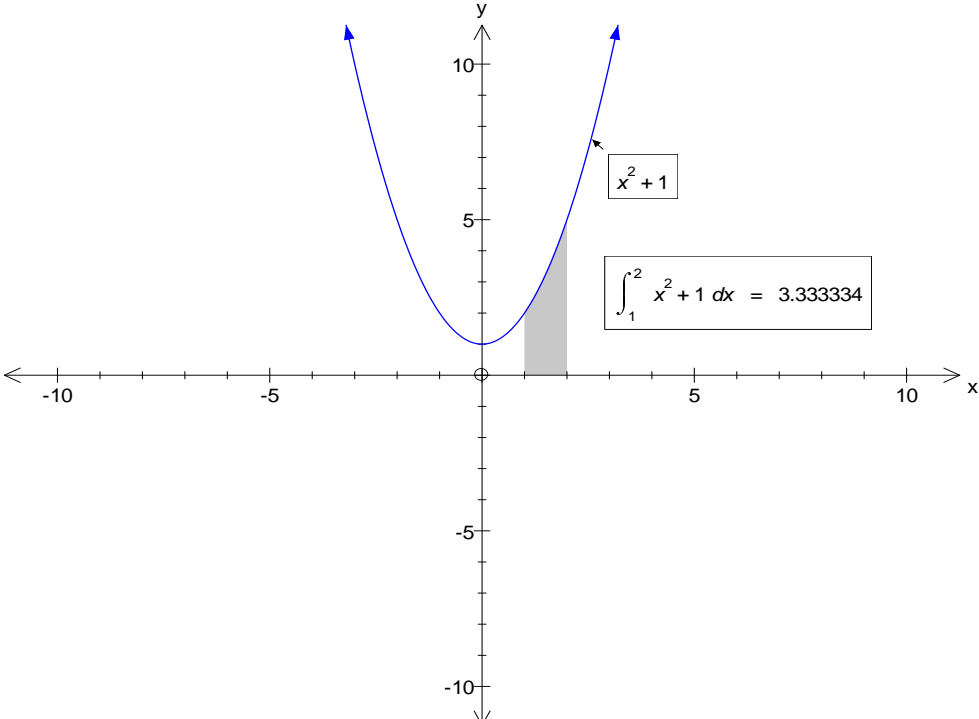
Now let us work out the area

We separate it into two areas A_1 (left side) and A_2 (right side of shape)

$$A_1 = \int_{-1}^0 x(x+1)(x-2)dx \quad \text{and} \quad A_2 = \int_0^2 x(x+1)(x-2)dx$$

- When we have two curves then in particular we need to be very careful, we must remember that to work out the area we need to subtract the curve on top from the curve beneath. The area enclosed is **ALWAYS CALCULATED AS THE UPPER CURVE MINUS THE LOWER CURVE REGARDLESS OF WHETHER ALL OR PART OF THE AREA IS BELOW THE X-AXIS**

More examples on calculating areas

<p>Find the area of the region enclosed by $y = x^2 + 1$, the x-axis, $x = 1$ and $x = 2$</p>	
<p>Sketch the graph and see what we are trying to find</p>	
<p>Now we can find the shaded area using integration</p>	$\int_1^2 (x^2 + 1) dx = \left(\frac{x^3}{3} + x \right)_1^2$ <p>Now we need to be very careful on doing this , remember to use brackets and watch out for the negative signs.</p> $\int_1^2 (x^2 + 1) dx = \left(\frac{2^3}{3} + 2 \right) - \left(\frac{1^3}{3} + 1 \right)$ <p>Now we need to work this out</p> $\int_1^2 (x^2 + 1) dx = \left(\frac{8}{3} + 2 \right) - \left(\frac{1}{3} + 1 \right)$ <p>And the answer is</p> $\int_1^2 (x^2 + 1) dx = \left(\frac{14}{3} \right) - \left(\frac{4}{3} \right)$ <p>So the answer is</p> $\int_1^2 (x^2 + 1) dx = \left(\frac{10}{3} \right) \text{ square units}$

Concept- Rules for antidifferentiaion

<p>RULE 1</p> $\int ax^n dx = \frac{ax^{n+1}}{n+1} + c, n \neq -1$ <p>We always add a real constant c since if we took the derivative this constant would disappear</p>	<p>Example:</p> $\text{Find } \int 3x^5 dx = \frac{3x^6}{6} + c$ $\text{Find } \int (3x^5 + 4x^2 + 5)dx = \frac{3x^6}{6} + \frac{4x^3}{3} + 5x + c$
<p>RULE 2</p> $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c, n \neq -1$	<p>Example</p> $\int (4x+1)^5 dx = \frac{(4x+1)^6}{4(6)} + c$ $\rightarrow \frac{(4x+1)^6}{24} + c$
<p>RULE 3</p> $\int e^{kx} dx = \frac{1}{k} e^{kx} + c$	<p>YOU CANNOT RAISE THE POWER BY 1; THIS IS NOT THE CORRECT METHOD FOR EXPONENTIALS. REMEMBER YOU CAN ONLY ANTIDIFFERENTIATE EXPONENTIAL FUNCTIONS TO BASE e. USE YOUR GRAPHICS CALCULATOR TO EVALUATE A DEFINITE INTEGRAL INVOLVING EXPONENTIAL FUNCTIONS OTHER THAN e</p>
<p>RULE 4</p> $\int (\cos kx) dx = \frac{1}{k} \sin kx + c$ $\int (\sin kx) dx = -\frac{1}{k} \cos kx + c$ $\int (\sec^2 kx) dx = \frac{1}{k} \tan kx + c$	
<p>RULE 5</p> $\int \frac{1}{x} dx = \log_e x + c$	<p>Watch out if we have the following equation</p> <p>Generally we have the following equation:</p> $\int \frac{1}{ax+b} dx = \frac{1}{a} \log_e (ax+b) + c$
<p>RULE 6</p> $\int \frac{f'(x)}{f(x)} dx = \log_e f(x) + c$	<p>Example on how the above rule works:</p> <p><u>Example 1</u></p>

	<p>Find $\int \frac{7}{5x+2} dx$</p> <p>Solution</p> $\int \frac{7}{5x+2} dx$ $\rightarrow 7 \int \frac{1}{5x+2} dx \quad \text{now we want } f'(x) = 5 \text{ since } f(x) = 5x+2$ <p>Divide by 5 so how equation will look like rule 6</p> $\frac{7}{5} \int \frac{1}{5x+2} dx$ $\rightarrow \frac{7}{5} \log_e(5x+2) + c$
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Various integration rules mixed – just to show you how many there are

<ul style="list-style-type: none"> properties of anti-derivatives and definite integrals: $\int ((af(x) \pm bg(x))) dx = a \int f(x) dx \pm b \int g(x) dx$ $\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$ $\int_a^b f(x) dx = - \int_b^a f(x) dx$ $\int_a^a f(x) dx = 0$ <p>Properties of anti-derivatives and definite integrals</p> $\int Af(x) dx = A \int f(x) dx$ $\int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx$ $\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$ $\int_a^b f(x) dx = - \int_b^a f(x) dx$	<p>Rules for anti-derivatives of x^n, where $n \in \mathbb{Q}$</p> <p>For $n \neq -1$,</p> $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ $\int ax^n dx = \frac{ax^{n+1}}{n+1} + C$ $\int (x-b)^n dx = \frac{(x-b)^{n+1}}{n+1} + C$ $\int a(kx-b)^n dx = \frac{a(kx-b)^{n+1}}{k(n+1)} + C.$ <p>For $n = -1$, i.e. x^{-1} or $\frac{1}{x}$, where $x \neq 0$,</p> $\int \frac{1}{x} dx = \log_e x + C$ $\int \frac{a}{kx} dx = \frac{a}{k} \log_e x + C$ $\int \frac{1}{x-b} dx = \log_e x-b + C$ $\int \frac{a}{k(x-b)} dx = \frac{a}{k} \log_e x-b + C$ $\int \frac{a}{kx-b} dx = \frac{a}{k} \log_e kx-b + C.$
<p>Rules for anti-derivative of e^{kx}</p> $\int e^x dx = e^x + C$ $\int ae^{kx} dx = \frac{a}{k} e^{kx} + C$	<p>Rules for anti-derivatives of $\cos(kx)$ and $\sin(kx)$</p> $\int \cos(x) dx = \sin(x) + C$ $\int a \cos(kx) dx = \frac{a}{k} \sin(kx) + C$

$\int e^{x-b} dx = e^{x-b} + C$ $\int ae^{k(x-b)} dx = \frac{a}{k} e^{k(x-b)} + C$ $\int ae^{kx-b} dx = \frac{a}{k} e^{kx-b} + C$	$\int \cos(x-b) dx = \sin(x-b) + C$ $\int a \cos k(x-b) dx = \frac{a}{k} \sin k(x-b) + C$ $\int a \cos(kx-b) dx = \frac{a}{k} \sin(kx-b) + C$ $\int \sin(x) dx = -\cos(x) + C$ $\int a \sin(kx) dx = -\frac{a}{k} \cos(kx) + C$ $\int \sin(x-b) dx = -\cos(x-b) + C$ $\int a \sin k(x-b) dx = -\frac{a}{k} \cos k(x-b) + C$ $\int a \sin(kx-b) dx = -\frac{a}{k} \cos(kx-b) + C$
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Skill builder on using the various rules

<p>RULE 1</p> $\int ax^n dx = \frac{ax^{n+1}}{n+1} + c, n \neq -1$ <p>We always add a real constant c since if we took the derivative this constant would disappear</p>	<p>Find the following:</p> $\int 6x^5 dx$ $\int (4x + 3x^2) dx$ $\int \left(\frac{6x + x^2}{7}\right) dx$
<p>RULE 2</p> $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c, n \neq -1$	$\int (3x+2)^2 dx$ $\int (6x-2)^5 dx$
<p>RULE 3</p> $\int e^{kx} dx = \frac{1}{k} e^{kx} + c$	$\int (4e^x) dx$ $\int (e^{5x}) dx$
<p>RULE 4</p> $\int (\cos kx) dx = \frac{1}{k} \sin kx + c$ $\int (\sin kx) dx = -\frac{1}{k} \cos kx + c$ $\int (\sec^2 kx) dx = \frac{1}{k} \tan kx + c$	$\int (\cos 4x) dx$ $\int (\sin 9x) dx$

<p>RULE 5</p> $\int \frac{1}{x} dx = \log_e x + c$ $\int \frac{1}{ax+b} dx = \frac{1}{a} \log_e(ax+b) + c$	$\int \left(\frac{1}{x}\right) dx$ $\int \left(\frac{8}{x}\right) dx$ $\int \left(\frac{1}{2x+3}\right) dx$
<p>RULE 6</p> $\int \frac{f'(x)}{f(x)} dx = \log_e f(x) + c$	$\int \left(\frac{2}{2x+3}\right) dx$ $\int \left(\frac{5}{5x-3}\right) dx$

Concept- given the anti-derivative find or evaluate the integral

Problems involving questions which ask you to differentiate a function and **hence antidifferentiate** a different function that you normally will not be able to do with one of the rules.

Solution

These questions ask you to differentiate a function, and then they **require** you to use this answer to work out a different integral. The difficult part is using the answer to do the second part.

Steps	
<p>If $y = x^4 + 2x^3$ find $\frac{dy}{dx}$ and hence find $\int (2x^3 + 3x^2) dx$</p>	
<p>Step 1: First find $\frac{dy}{dx}$</p>	$\frac{dy}{dx} = 4x^3 + 6x^2$
<p>Step 2: We need to re-organise this to start to look like the integral.</p> <p>Remember that the integral of $\frac{dy}{dx}$ is the original function.</p>	$\int (4x^3 + 6x^2) dx = x^4 + 2x^3 + c$ <p>Now notice if we take out a 2 from the left side it will look like the integral we want</p> $\int 2(2x^3 + 3x^2) dx = x^4 + 2x^3 + c$ <p>Now we can take the 2 outside the brackets</p> $2 \int (2x^3 + 3x^2) dx = x^4 + 2x^3 + c$ <p>Divide the LHS by 2 and do the same to the RHS</p> $\frac{2 \int (2x^3 + 3x^2) dx}{2} = \frac{x^4 + 2x^3 + c}{2}$ <p>Thus we finally manage to get it to look like</p> $\int (2x^3 + 3x^2) dx = \frac{x^4}{2} + \frac{2x^3}{2} + \frac{c}{2}$ $\int (2x^3 + 3x^2) dx = \frac{x^4}{2} + \frac{2x^3}{2} + c_1$

<p>Find $\frac{dy}{dx}$ of $x \cos x$ and hence find $\int 2x \sin x dx$</p>	
<p>Step 1- Find $\frac{dy}{dx}$ of $x \cos x$</p>	<p>$y = uv$</p> <p>Let $u = x \rightarrow \frac{du}{dx} = 1$</p> <p>Let $v = \cos x \rightarrow \frac{dv}{dx} = -\sin x$</p> $\frac{dy}{dx} = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$ $\frac{dy}{dx} = x(-\sin x) + (\cos x)1$ $\frac{dy}{dx} = -x \sin x + \cos x$
<p>Now we try to use the answer here and do the hence part of the question</p>	$\int (-x \sin x + \cos x) = x \cos x + c$ <p>Now separate the integral on the left</p> $\int (-x \sin x dx + \int \cos x dx = x \cos x + c$ $\rightarrow \int -x \sin x dx = x \cos x - \int \cos x dx + c$ $\rightarrow \text{divide by -1 both sides}$ $\rightarrow \int x \sin x dx = \sin x - x \cos x + c$ <p>Multiply by 2</p> $\int 2x \sin x dx = 2 \sin x - 2x \cos x + c$

Concept- Finding the area enclosed by graph and the x- axis

<p>Find the area enclosed by the graph $y = x(2-x)(x-3)$ and the x axis</p>	
<p>Step 1: Sketch the graph</p>	
<p>Step 2: Expand the brackets so that we can integrate the expression</p>	$x(2-x)(x-3) = -x^3 + 5x^2 - 6x$
<p>Step 3: Now we can set up the equation to find the area under the graph. Remember we will need to find the separate areas carefully.</p>	$\int_0^2 (-x^3 + 5x^2 - 6x) dx + \int_2^3 (-x^3 + 5x^2 - 6x) dx$ $= \left[\frac{-x^4}{4} + \frac{5x^3}{3} - \frac{6x^2}{2} \right]_0^2 + \left[\frac{-x^4}{4} + \frac{5x^3}{3} - \frac{6x^2}{2} \right]_2^3$ $= \left(\frac{-2^4}{4} + \frac{5 \cdot 2^3}{3} - \frac{6 \cdot 2^2}{2} \right) - \left(\frac{-0^4}{4} + \frac{5 \cdot 0^3}{3} - \frac{6 \cdot 0^2}{2} \right) + \left(\frac{-3^4}{4} + \frac{5 \cdot 3^3}{3} - \frac{6 \cdot 3^2}{2} \right) - \left(\frac{-2^4}{4} + \frac{5 \cdot 2^3}{3} - \frac{6 \cdot 2^2}{2} \right)$ $= \left(-2 \frac{2}{3} \right) + \left(\frac{5}{12} \right)$ $= 3 \frac{1}{12} \text{ sq units}$

NOTES FROM ABOVE EXAMPLE

When we find the area bounded between two graphs it is always the top graph – bottom graph

Examples:

<p>Find the area bounded between the x axis and the graph $y = x^2 - 3x$</p>	
<p>Step 1 Always sketch the graph</p> <p>Noticed that I used a graphics program to work out the area which is the shaded part of the graph.</p> <p>But we are going to do it properly using the ideas of top graph – bottom graph</p>	
<p>Step 2 To find the shaded area theory states to subtract top graph ($y = 0$) from bottom graph $y = x^2 - 3x$</p>	$\int_0^3 ((0) - (x^2 - 3x)) dx$ <p>Best to work the inside out separately so that we do not get confused</p> $\int_0^3 -(x^2 - 3x) dx$ <p>Which becomes the following when you multiply through with the negative</p> $\int_0^3 (-x^2 + 3x) dx$
<p>Step 3 Evaluate the integral</p>	$\left[\frac{-x^3}{3} + \frac{3x^2}{2} \right]_0^3$ <p>Now we put the $x = 3$ into the expression above and then subtract the $x = 0$ expression</p> $\left(\frac{-3^3}{3} + \frac{3 \times 3^2}{2} \right) - \left(\frac{-0^3}{3} + \frac{3 \times 0^2}{2} \right)$ <p>This becomes</p>

	$\left(\frac{-3^3}{3} + \frac{3 \times 3^2}{2}\right) - (0)$ <p>And the final answer is</p> $\left(\frac{-27}{3} + 27\right) = 18$
	This is the proper way of doing this. Some books just get to evaluate the integral and ignore the $y = 0$ (which we subtracted from $y = x^2 - 3x$)

Example 2

Find the area bounded by the two graphs $y = x$ and $y = x^2 - 4x$	
Step 1 – Sketch the two graphs and see what they look like	
Now we need to find the area bounded between the two graphs	$\int_0^5 ((x) - (x^2 - 4x)) dx$ <p>It is important to work the inside of the integral before actually working out the integral to avoid errors in the subtraction</p> $\int_0^5 (x - x^2 + 4x) dx$ <p>Which becomes</p> $\int_0^5 (-x^2 + 5x) dx$

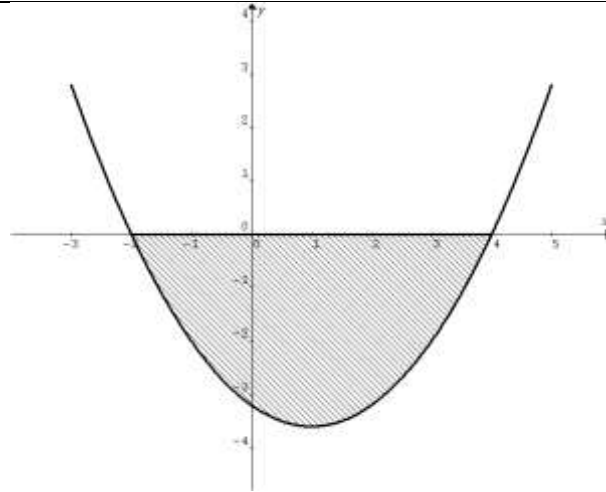
Now we take the integral	$\left[\frac{-x^3}{3} + \frac{5x^2}{2} \right]_0^5$
Now we apply the method	$\left(\frac{-5^3}{3} + \frac{5 \times 5^2}{2} \right) - \left(\frac{-0^3}{3} + \frac{5 \times 0^2}{2} \right)$
Which finally becomes	$\left(\frac{-125}{3} + \frac{125}{2} \right)$ Giving us the following answer $\frac{125}{6} \text{ sq units}$

Be careful of signs & Use graphics calculator to do the work for you by entering the equations in and find the values when the limit are put in.

**Various Worked examples to look over and see if you can do.
NO PARTICULAR ORDER**

1	Find an anti-derivative of $\frac{-3}{5(2x+1)^2}$. $\int \frac{-3}{5(2x+1)^2} dx = \int \frac{-3(2x+1)^{-2}}{5} dx = \frac{-3(2x+1)^{-1}}{5(-1)(2)} + C = \frac{3}{10(2x+1)} + C$ Choose any real value for C, usually 0 for convenience.
2	Find $F(x) = \int \frac{2}{3x-4} dx$ such that $F(1) = 1$. $F(x) = \frac{2}{3} \log_e 3x-4 + C, F(1) = \frac{2}{3} \log_e -1 + C = 1,$ $\therefore \frac{2}{3} \log_e (1) + C = 1, C = 1. \therefore F(x) = \frac{2}{3} \log_e 3x-4 + 1.$
3	Evaluate $\int_{-1}^1 (x^2 - 1)(x^2 + 1) dx$. $\int_{-1}^1 (x^2 - 1)(x^2 + 1) dx = \int_{-1}^1 (x^4 - 1) dx = \left[\frac{x^5}{5} - x \right]_{-1}^1$ $= \left(\frac{1}{5} - 1 \right) - \left(\frac{-1}{5} + 1 \right) = -\frac{8}{5}.$
4	Evaluate $\int_0^{\log_e 2} \frac{e^{2x} - e^{-2x}}{e^{2x}} dx$. $\int_0^{\log_e 2} \frac{e^{2x} - e^{-2x}}{e^{2x}} dx = \int_0^{\log_e 2} (1 - e^{-4x}) dx = \left[x + \frac{1}{4} e^{-4x} \right]_0^{\log_e 2}$ $= \left(\log_e 2 + \frac{1}{4} e^{-4 \log_e 2} \right) - \left(0 + \frac{1}{4} e^0 \right) = \log_e 2 + \frac{1}{4} \times \frac{1}{16} - \frac{1}{4}$ $= \log_e 2 - \frac{15}{64}.$

5	<p>Evaluate $\int_0^1 \frac{x^2+1}{x+1} dx$.</p> $\int_0^1 \frac{x^2+1}{x+1} dx = \int_0^1 \left(x-1 + \frac{2}{x+1} \right) dx = \left[\frac{x^2}{2} - x + 2 \log_e(x+1) \right]_0^1$ $= \left(\frac{1}{2} - 1 + 2 \log_e 2 \right) - (0) = 2 \log_e 2 - \frac{1}{2}.$
Integration by recognition	
6	<p>Find $\int \frac{1}{\cos^2 x} dx$.</p> $\int \frac{1}{\cos^2(x)} dx = \int \sec^2(x) dx = \tan(x) + C$ <p>by recognising that $\frac{d}{dx}(\tan(x)) = \sec^2(x)$.</p>
7	<p>Find the derivative of $x \log_e x$. Hence find the anti-derivative of $\log_e x$.</p> <p>Let $y = x \log_e x$, apply the product rule to obtain $\frac{dy}{dx} = (\log_e x)(1) + (x)\left(\frac{1}{x}\right) = \log_e x + 1$,</p> $\therefore \log_e x = \frac{dy}{dx} - 1. \int \log_e x dx = \int \left(\frac{dy}{dx} - 1 \right) dx = \int \frac{dy}{dx} dx - \int 1 dx$ $= y - x + C = x \log_e x - x + C.$
8	<p>Given $y = xe^{-2x}$, find $\frac{dy}{dx}$ and hence evaluate $\int_0^1 xe^{-2x} dx$.</p> $y = xe^{-2x}, \therefore \frac{dy}{dx} = (x)(-2e^{-2x}) + (1)(e^{-2x}) = -2xe^{-2x} + e^{-2x}$ $\therefore xe^{-2x} = \frac{1}{2} \left(e^{-2x} - \frac{dy}{dx} \right), \therefore \int_0^1 xe^{-2x} dx = \int_0^1 \frac{1}{2} \left(e^{-2x} - \frac{dy}{dx} \right) dx$ $= \left[\frac{1}{2} \left(\frac{e^{-2x}}{-2} - xe^{-2x} \right) \right]_0^1 = \frac{1}{2} \left(\frac{e^{-2}}{-2} - e^{-2} \right) - \frac{1}{2} \left(\frac{1}{-2} - 0 \right)$ $= \frac{1}{4} - \frac{3e^{-2}}{4} = \frac{1}{4} (1 - 3e^{-2}).$
Finding area under a curve by integration	
9	<p>Find the area of the region enclosed by the x-axis and $y = 0.4(x+2)(x-4)$.</p>



The left and right extremes of the region are the x -intercepts, $x = -2$ and $x = 4$ respectively, given by the linear factors.

Expand $y = 0.4(x+2)(x-4)$ to obtain $y = 0.4(x^2 - 2x - 8)$.

$$\begin{aligned} \text{Area} &= \left| \int_{-2}^4 0.4(x^2 - 2x - 8) dx \right| = \left| \left[0.4 \left(\frac{x^3}{3} - x^2 - 8x \right) \right]_{-2}^4 \right| \\ &= \left| 0.4 \left(\frac{4^3}{3} - 4^2 - 8(4) \right) - 0.4 \left(\frac{(-2)^3}{3} - (-2)^2 - 8(-2) \right) \right| = 14.4 \end{aligned}$$

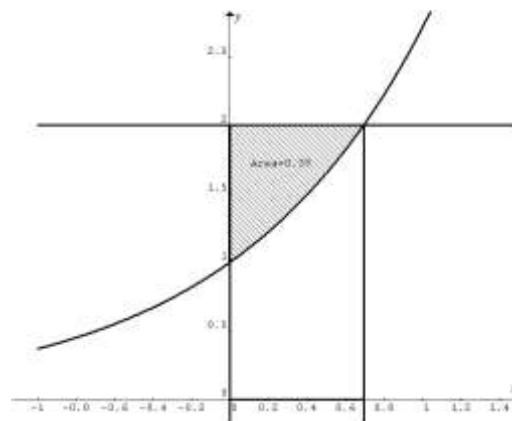
Find the area of the region bounded by the y -axis, the line $y = 2$ and the curve $y = e^x$.

For the curve $y = e^x$, when $y = 2$, $e^x = 2$, $\therefore x = \log_e 2$.

Area of the required region = area of the rectangular region – the area of the region under $y = e^x$ from $x = 0$ to $x = \log_e 2$

$$\begin{aligned} &= 2 \log_e 2 - \int_0^{\log_e 2} e^x dx = 2 \log_e 2 - [e^x]_0^{\log_e 2} = 2 \log_e 2 - (2 - 1) \\ &= 0.39 \end{aligned}$$

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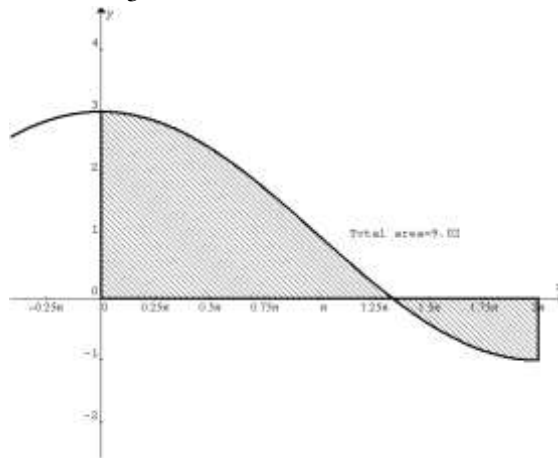
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4 Find the area of the region(s) bounded by the curve $y = 2 \cos \frac{x}{2} + 1$, the x -axis, the lines $x = 0$ and $x = 2\pi$.

Sketch the graph first. The curve crosses the x -axis. Find this x -intercept. Let $y = 0$, $2 \cos \frac{x}{2} + 1 = 0$,

$$\cos \frac{x}{2} = -\frac{1}{2}, \therefore \frac{x}{2} = \frac{2\pi}{3}, \therefore x = \frac{4\pi}{3}.$$

$$\begin{aligned} \text{Area} &= \left| \int_0^{\frac{4\pi}{3}} \left(2 \cos \frac{x}{2} + 1 \right) dx \right| + \left| \int_{\frac{4\pi}{3}}^{2\pi} \left(2 \cos \frac{x}{2} + 1 \right) dx \right| \\ &= \left| \left[4 \sin \frac{x}{2} + x \right]_0^{\frac{4\pi}{3}} \right| + \left| \left[4 \sin \frac{x}{2} + x \right]_{\frac{4\pi}{3}}^{2\pi} \right| \\ &= \left| 4 \sin \frac{2\pi}{3} + \frac{4\pi}{3} \right| + \left| \left(4 \sin \pi + 2\pi \right) - \left(4 \sin \frac{2\pi}{3} + \frac{4\pi}{3} \right) \right| \\ &= 2\sqrt{3} + \frac{4\pi}{3} + \left| 2\pi - 2\sqrt{3} - \frac{4\pi}{3} \right| \\ &= 2\sqrt{3} + \frac{4\pi}{3} + \left| \frac{2\pi}{3} - 2\sqrt{3} \right| \\ &= 2\sqrt{3} + \frac{4\pi}{3} + 2\sqrt{3} - \frac{2\pi}{3} = 4\sqrt{3} + \frac{2\pi}{3}. \end{aligned}$$



Find the area of the region enclosed by $y = x^2 + 2x - 1$ and $y = -x^2 + 2x + 1$.

Solve the two equations simultaneously to find the x -coordinates of the intersections.

$$x^2 + 2x - 1 = -x^2 + 2x + 1, \therefore 2x^2 = 2, \therefore x = \pm 1.$$

$$\begin{aligned} \text{Area} &= \left| \int_{-1}^1 \left((x^2 + 2x - 1) - (-x^2 + 2x + 1) \right) dx \right| \\ &= \left| \int_{-1}^1 (2x^2 - 2) dx \right| = \left| \left[\frac{2x^3}{3} - 2x \right]_{-1}^1 \right| = \left| \left(\frac{2}{3} - 2 \right) - \left(-\frac{2}{3} + 2 \right) \right| = \frac{8}{3}. \end{aligned}$$

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