Updated-10 Feb 2011

# Graphs

## Preliminary notes on graphs for chapter 3/4

A quick summary of the main concepts regarding the different type of graphs for Yr 12 Methods

# Graphs of various functions

In this section we will discuss how we can plot some simple graphs involving functions of x. In particular we will look at the power functions + polynomial functions

## Quick Background

Quite often we tend to forget the work we did in previous years and as a result get into strive when we are called upon to remember some common facts, so here I will review some of the facts from previous years

### Straight line graphs

These graphs are normally expressed as follows $y = mx + c$, where m = is the gradient of the line and c = Y intercept
Sometimes these graphs can also be expressed as $ax + by = c$ , where a, b, and c are real numbers.

Typical graph

Sketch the graph of $y = 2x + 3$

Many ways to sketch this graph

1) Use the graphics calculator
2) Put values of x and find the corresponding values of y and plot them
3) Find the X-intercept and Y-intercept and then draw a line between them

Let's use method 3 as a starting point

First let us set x = 0 and substitute this into the equation we get the following

$$y = 2x + 3$$
$$y = 2(0) + 3$$
$$y = 3$$

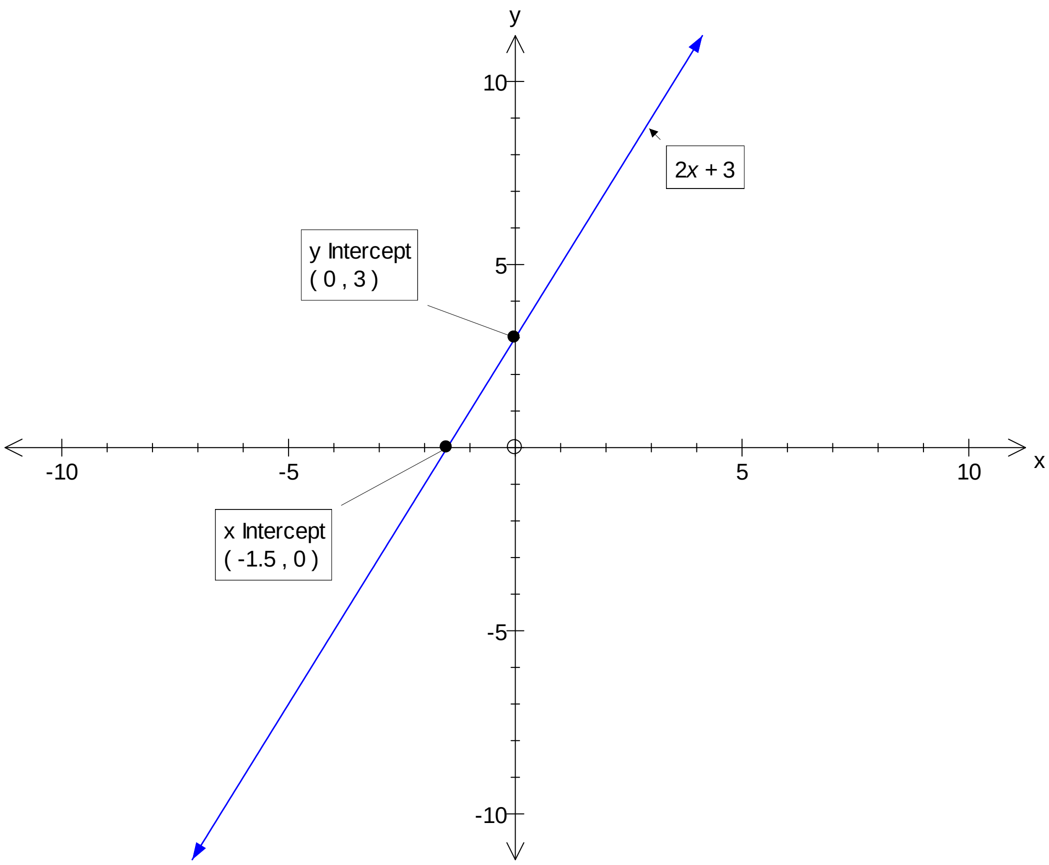
This gives us the Y intercept, so the point is then ( 0,3)

Now let us set y = 0 and substitute this into the equation and solve

$$y = 2x + 3$$
$$0 = 2x + 3$$
$$2x + 3 = 0$$
$$2x = -3$$
$$\frac{2x}{2} = \frac{3}{2}$$
$$x = 1.5$$

So then we obtain the X – intercept and the coordinates are ( 1.5, 0)

Now we sketch the following graph using these two intercepts and we get the following




What does the gradient mean?

It is basically the slope of the graph. If it slopes to the right we say it has a positive slope, whereas if it slopes to the left we say it has a negative slope.

Important formulae for straight line equations

| Important ideas regarding straight lines | |
|---|---|
| Normal equation of straight line | $y = mx + c$ |
| Finding the gradient between two points $(x_1, y_1)$ and $(x_2, y_2)$ | $m = \frac{(y_2) - (y_1)}{(x_2) - (x_1)}$ |
| Finding equation if it passes through a point $(x_1, y_1)$ and has a gradient of m | $y - y_1 = m(x - x_1)$ |
| Relationship between parallel gradient and perpendicular gradient of two straight lines | $M_p M_\perp = -1$ where $M_p$ stands for parallel gradient and $M_\perp$ stands for perpendicular gradient |

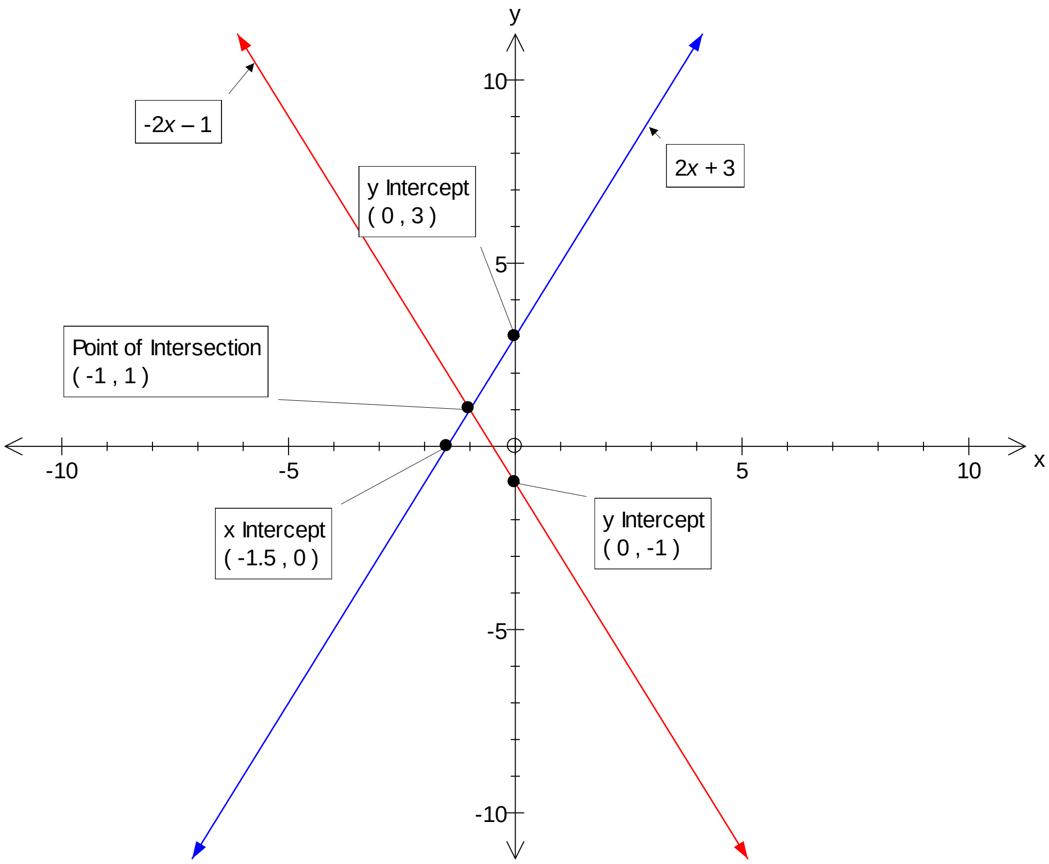
**Solving simultaneous equations using straight lines**

Basically when you have to solve simultaneous equations what it really means is trying to find a common point that is shared by both equations

Example: Solve the following two graphs for a common solution

$$y = 2x + 3 \text{ And } y = -2x - 1$$

Let us plot the two graphs and see what happens




So the solution to both of these graphs is the point where they intercept, namely in this case at x = -1

## Quadratic Graphs

These graphs follow the following equation $y = ax^2 + bx + c$, where a, b and c are real numbers

How do we sketch quadratic graphs?

They are a little bit trickier than linear function but there are a variety of trusted methods

**Method-1**- Just selected a variety of values of x and find the corresponding value of y and plot them. This takes time but at least it is fail safe.

**Method-2**-Use a graphic calculator and it will do it for you. Good method but in an exam where you are not allowed to use a graphics calculator it will not help.

**Method-3-**Factorize the equation if it is possible and find the y intercept should give a rough sketch of the quadratic graph.

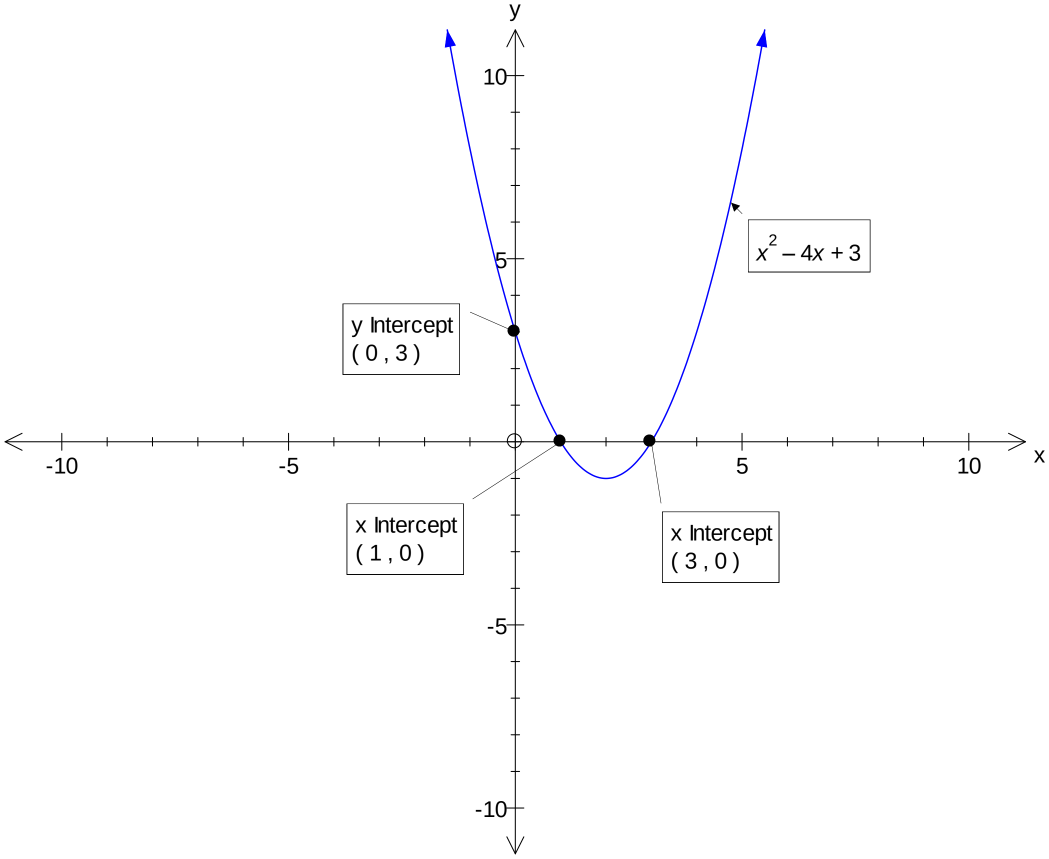
**Method-4-**Complete the square as this will give us the turning point and again give us a good way to sketch the graph.

**Method -5-** Use the formula to find the roots of the quadratic equation and sketch it

We will need to be able to able to use all the above methods in sketching quadratic equations.

Let us review each of the methods above

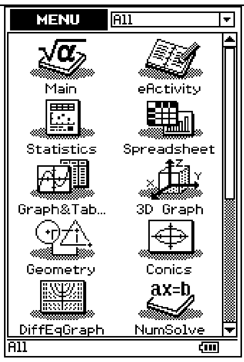
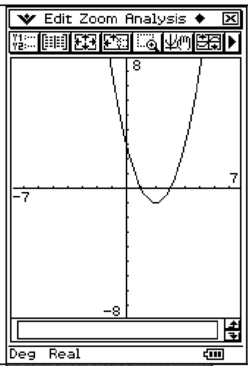
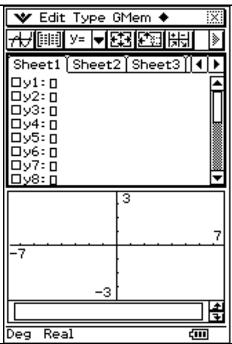
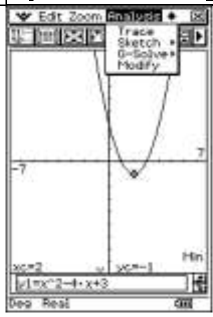
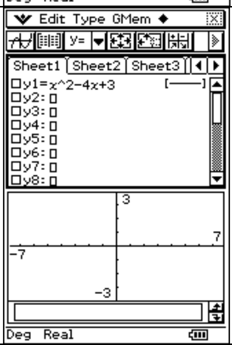
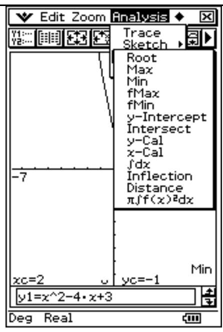
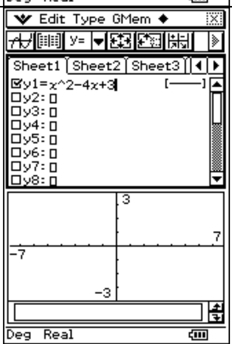
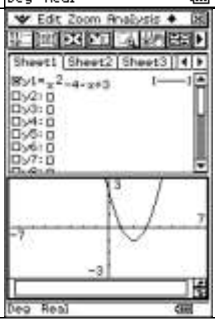
**Method-1-Sketching graph using values gives us a graph as follows:**




**Method-2-Using the Casio classpad calculator**

Sketch $y = x^2 - 4x + 3$

Step in using Casio classpad

| | | | |
|---|---|---|---|
| 1-Start classpad |  | 6-Now press the rezise screen |  |
| 2-Press Graph& Tab and you will get the following |  | 7-Now press analysis we can find the answers to various questions<br>Press G-solve |  |
| 3-Enter the equation into the classpad |  | 8-<br>After pressing G-solve we can find the root( x intercepts) ,Min,Max etc |  |
| 4-Tick the box next to the graph |  | 9- Pressing G-Solve we can find answers to various questions. | Roots x =1 and x = 3<br>Turning points are x =2 y= -1<br>Y intercept y = 3 |
| 5-Now press sketch |  | | |

**Method-3-Factorizing / Using the formula**

This method does not always work so it actually pays to find the determinant first to see if the equation can be factorized.

Remember the general equation is the following: $y = ax^2 + bx + c$

Now the general solution is given by the following equation $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

The determinant of a quadratic is what is inside the square root brackets and is given a special symbol $\Delta$

$\Delta = b^2 - 4ac$

Now looking at the table below we find the determinant has a few special conditions

| $\Delta$ | Meaning | Graphical meaning |
|---|---|---|
| $\Delta = 0$ | One solution exist | Graph touches x axis once |
| $\Delta > 0$ | Two solutions exists | Graph crosses the x axis twice |
| $\Delta < 0$ | No solution exists | Graph does not cross x axis at all |

So let us return back to the original graph and see it stacks up this time using the formula

$y = x^2 - 4x + 3$

What I normally do is compare it to the standard equation and note the values of a, b and c

| | | | |
|---|---|---|---|
| Y | $ax^2$ | $+bx$ | $+c$ |
| Y | $1x^2$ | $-4x$ | $+3$ |
| | $a = 1$ | $b = -4$ | $c = 3$ |
| | | | |

Now find $\Delta$ to see if solutions actually exists (that does not mean we cannot sketch it if no solutions exists, just that it does not cross the x-axis!)

$\Delta = b^2 - 4ac = (-4)^2 - 4(1)(3) = 16 - 12 = 4$ and since the determinant is greater than zero it means it crosses the x axis twice.

Now lets us use the big formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to find the values of x, we need to be careful here.

We have already found the determinant which is the part in the square root sign so we need to put it into the formula and proceed as normal

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-4) \pm \sqrt{4}}{2(1)} = \frac{4 \pm \sqrt{4}}{2} = \frac{4 \pm 2}{2}$$

So we can two values for x, namely the following

$$x = \frac{4+2}{2} \text{ or } x = \frac{4-2}{2}$$

$$x = \frac{6}{2} \text{ or } x = \frac{2}{2}$$

$x = 3$ or $x = 1$

This gives us the roots where the graph crosses the x-axis. In fact we can actually factorise it also as manipulating these two solutions gives us the following:

$$(x-3)(x-1) = x^2 - 4x + 3 = y$$

This method does not give us the turning point but because of symmetry you can guess that the turning point is located always half way between the two roots.

A simple formula is given as follows $\left(\frac{-b}{2a}, \frac{-\Delta}{4a}\right)$, this gives us the turning point of the parabola

Testing the formula yields the following results: $\left(\frac{-b}{2a}, \frac{-\Delta}{4a}\right) = \left(\frac{-(-4)}{2(1)}, \frac{-4}{4(1)}\right) = \left(\frac{4}{2}, \frac{-4}{4}\right) = (2, -1)$ which is exactly what we expected

Various other facts about quadratic equations

| Facts about quadratic equations | |
|---|---|
| General equation of a quadratic equation | $y = ax^2 + bx + c$ |
| If a > 0 then parabola opens upwards<br>If a < 0 then parabola opens downwards | The coefficient a controls the speed on increase or decrease of the quadratic function from the vertex<br><br>Bigger positive a makes the function increase faster and the graph appear more closed |
| The coefficient c controls the height of the parabola more specifically it is the point where it crosses the y axis | $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ |
| The coefficient b alone is the declivity of the parabola as it crosses the y-axis | |
| Finding the roots of a quadratic equation using the formula on the left | $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{\Delta}}{2a}$ |
| Finding the determinant, | $\Delta = b^2 - 4ac$ |

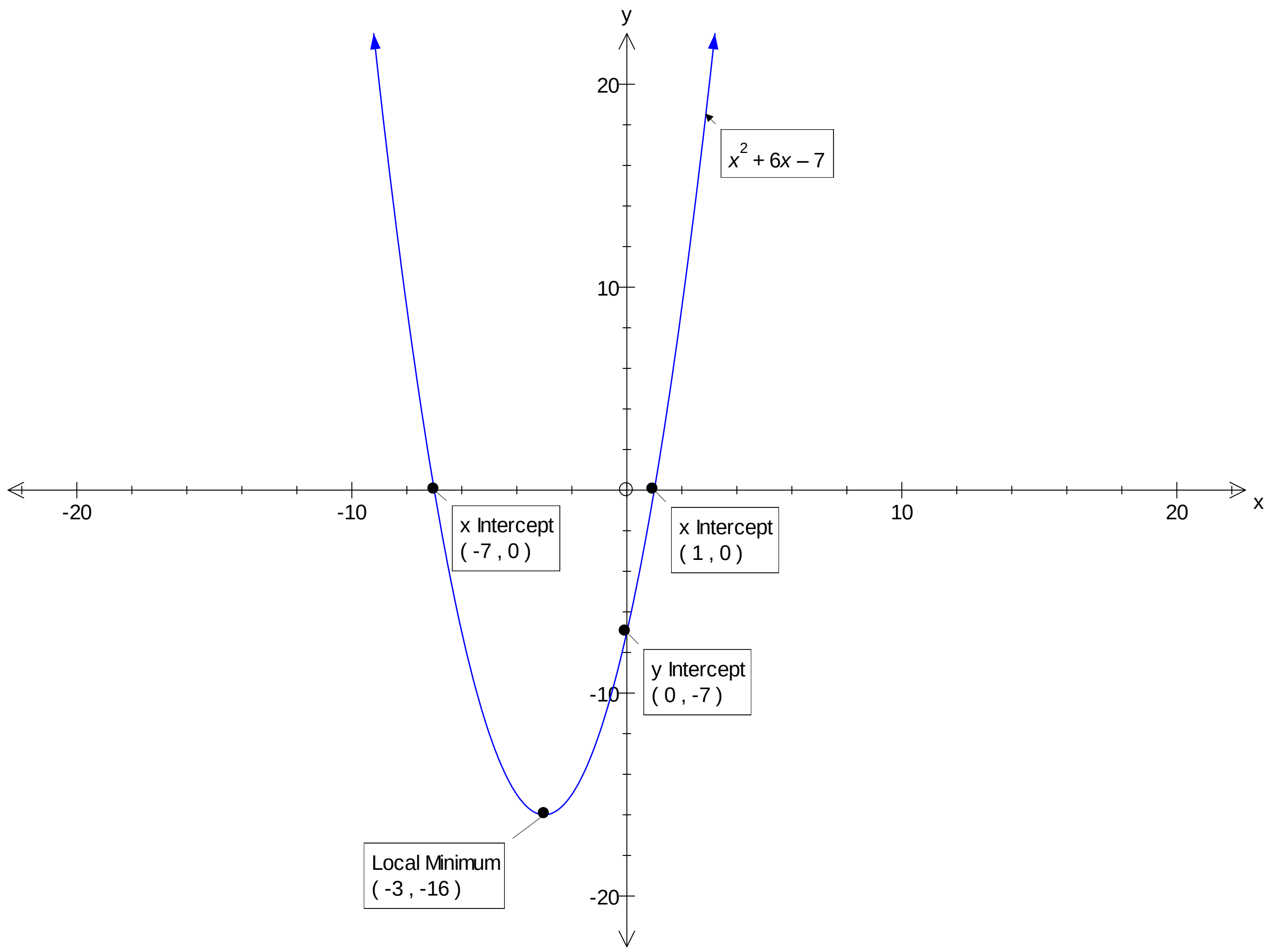
**Complete the square**

Let's look at the process of completing the square

| | |
|---|---|
| Start with the original quadratic | $x^2 + 6x - 7 = 0$ |
| Move the loose number over to the other side | $x^2 + 6x = 7$ |
| Take half the x term ( that is divide it by two, and do not forget the sign).<br>Square the answer<br>Add this to both sides of the equation | $x^2 + 6x + \left(\frac{6}{2}\right)^2 = 7 + \left(\frac{6}{2}\right)^2$<br><br>$x^2 + 6x + (3)^2 = 7 + (3)^2$<br><br>$x^2 + 6x + 9 = 7 + 9$<br>$x^2 + 6x + 9 = 16$ |
| Convert the left hand side to the squared form | $(x+3)^2 = 16$ |
| Now square root both sides and remember to include $\pm$ | $(x+3) = \pm\sqrt{16}$ |

| | |
|---|---|
| | $(x+3)=\pm 4$ |
| Now put it into simpler forms showing the two equations | $x+3=+4$ and $x+3=-4$ |
| Now solve for x and simplify the equations | $x+3=4 \quad x+3=-4$<br>$x=4-1$ and $x=-4-1$<br>$x=3 \quad x=-5$ |
| | |
| We could express this as a equation and complete the square | $y=x^2+6x-7$ |
| | $y=x^2+6x+\left(\frac{6}{2}\right)^2-7-\left(\frac{6}{2}\right)^2$ |
| | $y=x^2+6x+9-7-9$ |
| This is the vertex form of a quadratic equation<br><br>It actually shows the turning point which is located at $x=-3$ and $y=-16$ | $y=(x+3)^2-16$ |

Let's us plot the graph to have a look at it



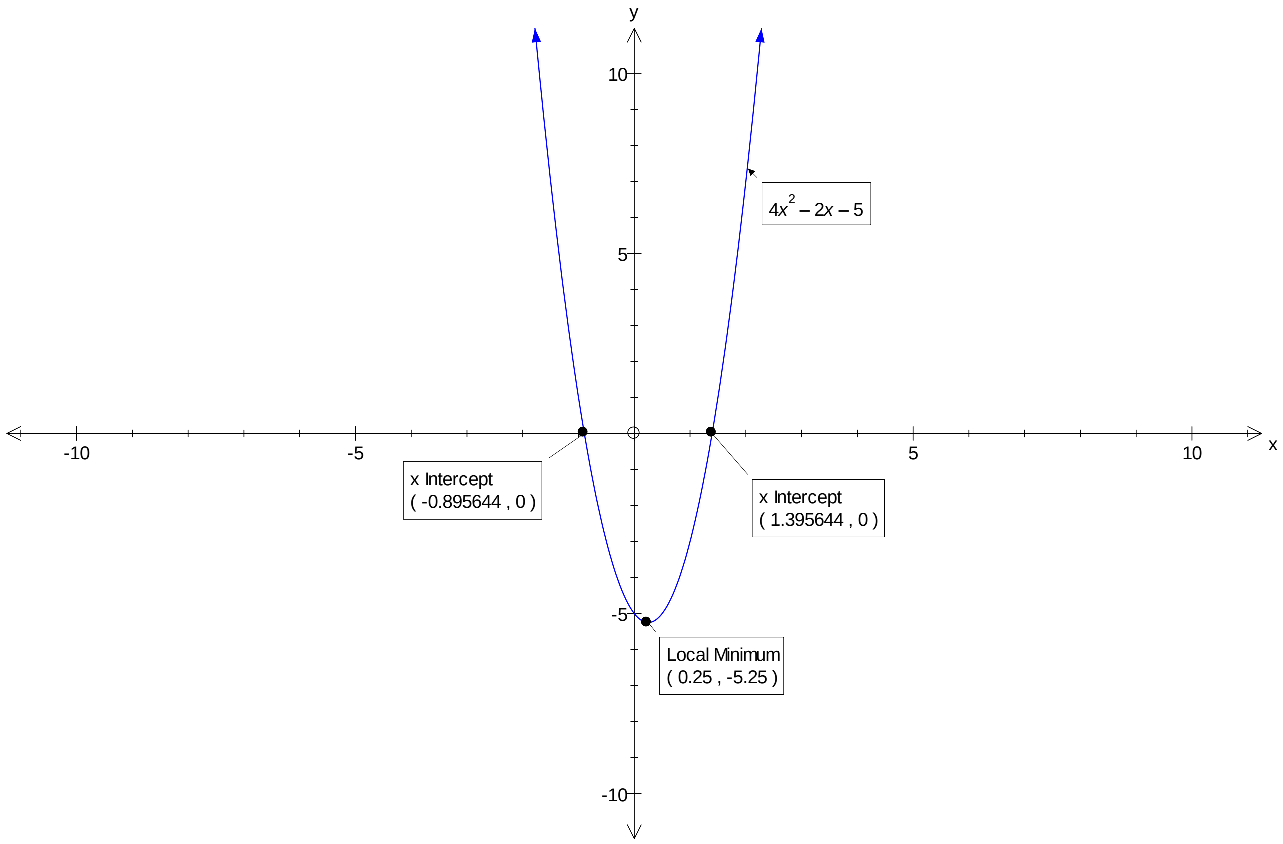

What happens when we have a number in front of the original equation?

Example $y=4x^2-2x-5$

Let us see if we can actually solve this equation without using the formula, that is using the determinant and the big formula.

| | |
|---|---|
| Start with the original quadratic | $4x^2-2x-5=0$ |
| Take out the 4 from both sides | $4(x^2-\frac{2}{4}x-\frac{5}{4})=0$ |
| Focus on the middle inside the brackets<br>Take half the x term ( that is divide it by two, and do not forget the sign).<br>Square the answer<br>Add this to both sides of the equation | $x^2-\frac{2}{4}x-\frac{5}{4}=0$<br>$x^2-\frac{1}{2}x-\frac{5}{4}=0$<br>$x^2-\frac{1}{2}x=\frac{5}{4}$<br>$x^2-\frac{1}{2}x+\left(\frac{-\frac{1}{2}}{2}\right)^2=\frac{5}{4}+\left(\frac{-\frac{1}{2}}{2}\right)^2$<br>$x^2-\frac{1}{2}x+\left(\frac{-1}{4}\right)^2=\frac{5}{4}+\left(\frac{-1}{4}\right)^2$<br>$x^2-\frac{1}{2}x+\frac{1}{16}=\frac{5}{4}+\frac{1}{16}$<br>$\left(x-\frac{1}{4}\right)^2=\frac{20}{16}+\frac{1}{16}$<br>$\left(x-\frac{1}{4}\right)^2=\frac{21}{16}$ |
| Now square root both sides and remember to include $\pm$ | $\left(x-\frac{1}{4}\right)=\pm\sqrt{\frac{21}{16}}$<br>$\left(x-\frac{1}{4}\right)=\pm\frac{\sqrt{21}}{4}$ |
| Now put it into simpler forms showing the two equations and we can find real answers to both of these equations | $x=\frac{1}{4}\pm\frac{\sqrt{21}}{4}$<br>$x=\frac{1}{4}+\frac{\sqrt{21}}{4}$ or $x=\frac{1}{4}-\frac{\sqrt{21}}{4}$ |
| We could have put the equation into vertex form and expressed it in such a way so to see the turning point of this parabola<br><br>$y=4x^2-2x-5$ | |

| | |
|---|---|
| Minimum: Y min=-5.25 for x = 0.25<br><br>Root: x = -0.8956439237<br><br>Root: x = 1.395643924<br><br>Turning point can be seen from the equation in vertex form $x = 0.25$ and $y = -5.25$ | $y = 4x^2 - 2x - 5$<br>$y = 4\left(x^2 - \frac{2x}{4} - \frac{5}{4}\right)$<br>$y = 4\left(x^2 - \frac{1x}{2} - \frac{5}{4}\right)$<br>$y = 4\left(x^2 - \frac{1x}{2} + \frac{1}{16} - \frac{5}{4} - \frac{1}{16}\right)$<br>$y = 4\left(\left(x - \frac{1}{4}\right)^2 - \frac{20}{16} - \frac{1}{16}\right)$<br>$y = 4\left(\left(x - \frac{1}{4}\right)^2 - \frac{21}{16}\right)$<br>$y = 4\left(x - \frac{1}{4}\right)^2 - \frac{84}{16}$<br>$y = 4\left(x - \frac{1}{4}\right)^2 - 5.25$ |



**Sketching Hyperbolas**

Let's us consider graphs of this form $y = \frac{a}{x}$ where a is any real number

To help us we will look at a few examples so we can see how to sketch these types of graphs

| Example | Hyperbola |
|---|---|
| 1 | $y = \frac{3}{x}$ |
| 2 | $y = \frac{3}{x-1}$ |
| 3 | $y = \frac{3}{x-1} + 2$ |

Example 1: $y = \frac{3}{x}$

Analysis

These families of graphs have normally two asymptotes, lines which the graph approaches but never actually touches. One of the asymptotes is on the x-axis and the other is on the y-axis

How do we find the asymptotes?

**For the Y asymptote let us investigate and see what happens when x gets very small**

| $x = 1$ | $x = 0.1$ | $x = 0.001$ | $x = 0.000001$ | $x \to +0$ |
|---|---|---|---|---|
| $y = \frac{3}{1} = 3$ | $y = \frac{3}{0.1} = 30$ | $y = \frac{3}{0.001} = 3000$ | $y = \frac{3}{0.000001} = 3000000$ | $y \to +\infty$ |
| Notice what happens as x gets **smaller** | Y is getting larger | Y is getting even larger as x is getting smaller | | So it is safe to say that y will approach a very large number as x approaches 0 from the positive side of x |

Now if we repeat this for negative values of x we would get the same things except this time the values would be negative

| $x = -1$ | $x = -0.1$ | $x = -0.001$ | $x = -0.000001$ | $x \to -0$ |
|---|---|---|---|---|
| $y = \frac{3}{1} = -3$ | $y = \frac{3}{0.1} = -30$ | $y = \frac{3}{0.001} = -3000$ | $y = \frac{3}{0.000001} = -3000000$ | $y \to +\infty$ |
| Notice what happens as x **gets larger** | Y is getting smaller | Y is getting even smaller as x is getting larger | | So it is safe to say that y will approach a very large negative number as x approaches 0 from the negative side of x |

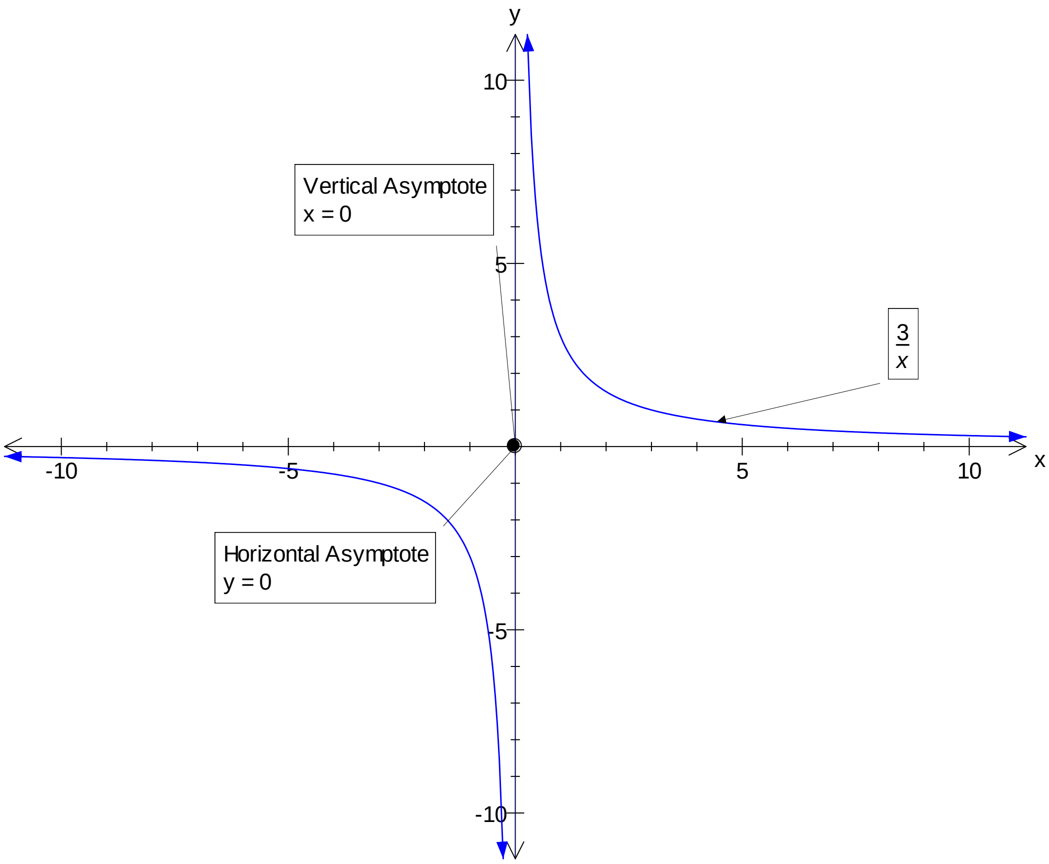
**For the X asymptote let us investigate what happens to x when it becomes large**

| $x=1$ | $x=10$ | $x=1000$ | $x=10000$ | $x\to+\infty$ |
|---|---|---|---|---|
| $y=\frac{3}{1}=3$ | $y=\frac{3}{10}=0.3$ | $y=\frac{3}{1000}=0.003$ | $y=\frac{3}{10000}=0.0003$ | $y\to+0$ |
| Notice what happens as x gets **smaller** | Y is getting smaller | Y is getting even smaller as x is getting larger | X is getting larger but Y is getting smaller | So it is safe to say that y will approach 0 as x approaches a large positive number |

What happens when the opposite occurs, i.e. x becomes very small

| $x=-1$ | $x=-10$ | $x=-1000$ | $x=-10000$ | $x\to-\infty$ |
|---|---|---|---|---|
| $y=\frac{3}{-1}=-3$ | $y=\frac{3}{-10}=-0.3$ | $y=\frac{3}{-1000}=-0.003$ | $y=\frac{3}{-10000}=-0.0003$ | $y\to-0$ |
| Notice what happens as x gets **smaller** | X is getting smaller but y is getting larger | Y is approaching zero from the negative side | | So it is safe to say that y will approach 0 as x approaches a large negative number |

How does this information translate into a graph then?




This is a typical hyperbola. In the above example we see it does not intercept the x-axis or the y-axis

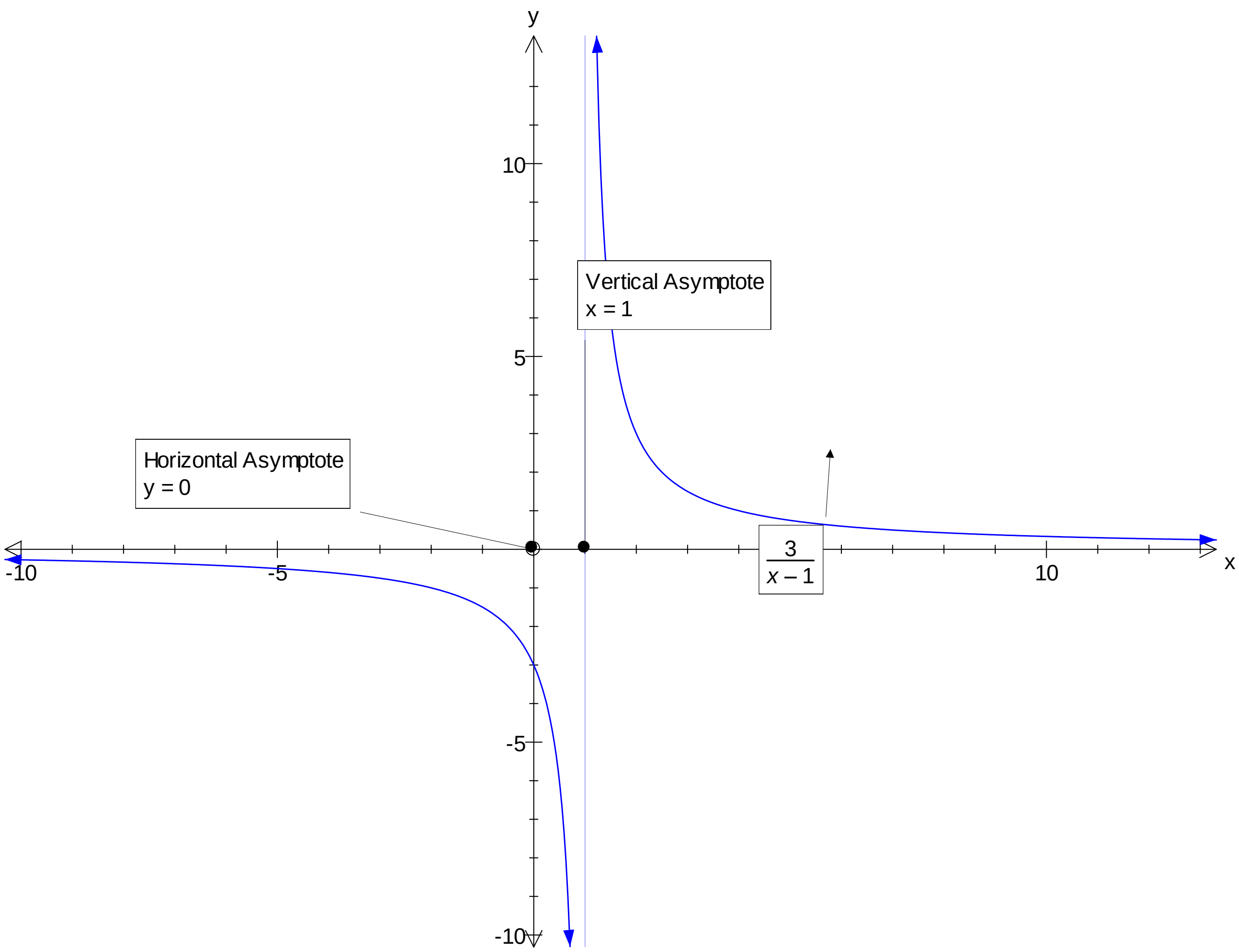
Now let us move onto the next example

Example 2: $y = \frac{3}{x-1}$

Analysis: The same as before, but pay attention to the denominator of the equation,

When x becomes 1 then y becomes undefined! ( $y = \frac{3}{x-1} = \frac{3}{0} = \text{not defined}$ )

So we have an X asymptote at x = 1
We still have an y asymptote at y = 0




Example 3: $y = \frac{3}{x-1} + 2$

Analysis
If x approaches a very large number then y approaches 2, therefore y = 2 is an asymptote
If x approaches 1 then y approaches will become undefined , therefore x =1 is an asymptote

We need to work out the x-intercepts and the y-intercepts

X-intercepts- set y = 0 and solve the resulting equation

$$0=\frac{3}{x-1}+2$$

$$\frac{3}{x-1}=-2$$

$$3=-2(x-1)$$

$$-2x+1=3$$

$$-2x=2$$

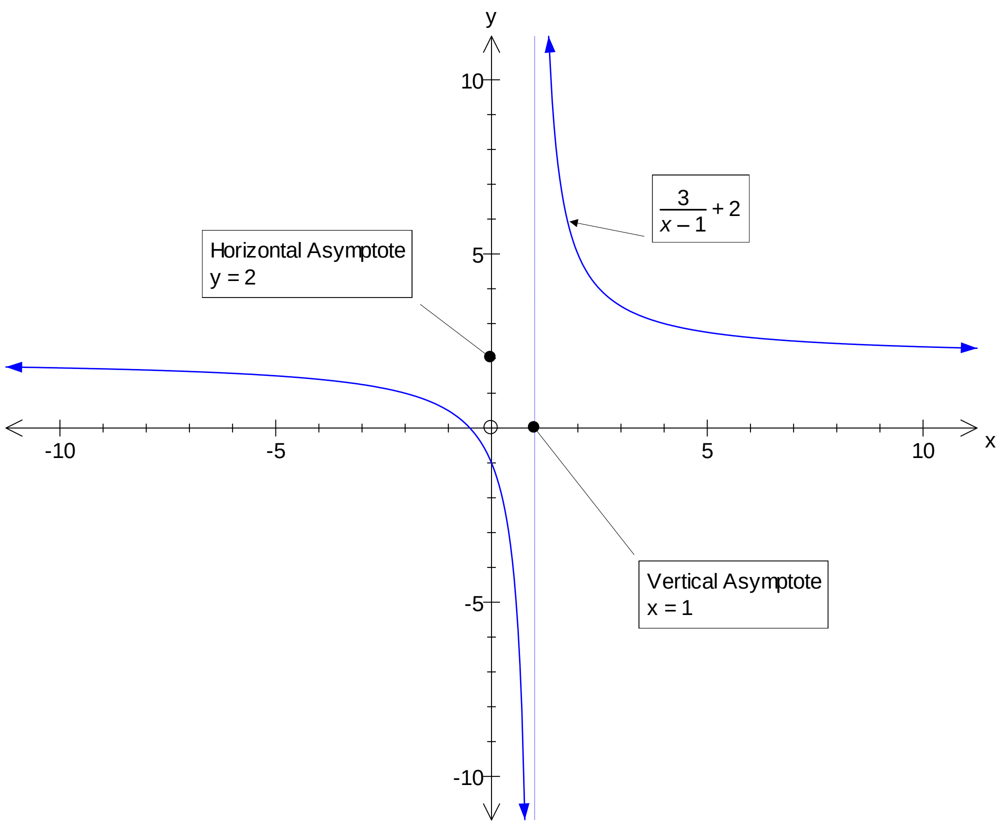
$$x=\frac{2}{-2}=-1$$

So (-1, 0) is the x intercept

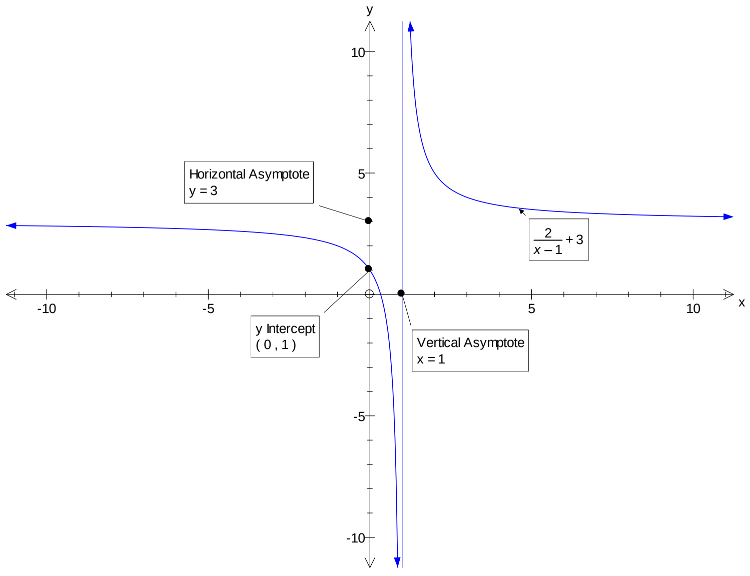
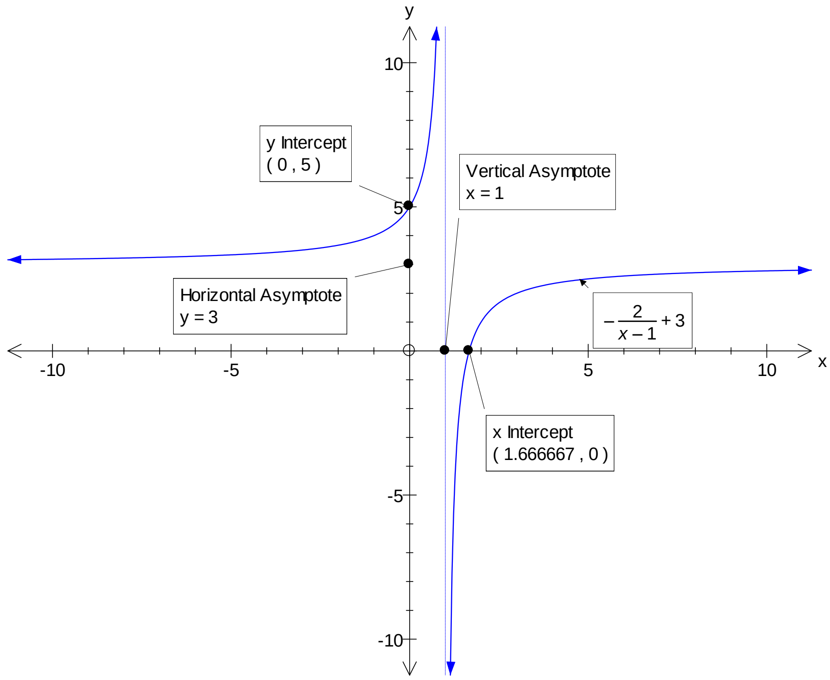
Y intercept set x = 0

$$y=\frac{3}{x-1}+2=\frac{3}{-1}+2=-1$$

Y intercept is (0,-1)



General facts about hyperbolas

| General Facts about Hyperbolas | |
|---|---|
| | $y = \frac{a}{x-h} + k$ |
| Vertical asymptote, x, | $x = h$ |
| Horizontal asymptote | $y = k$ |
| x-intercept | $(h - \frac{a}{k}, 0)$ |
| y-intercept | $(0, k - \frac{a}{h})$ |
| a > 0 |  |
| a < 0<br>( notice the graph is in the opposite quadrants) |  |

**Be careful:**

Sometimes the graphs are not written in the normal form and you might have to manipulate them a little bit

| Form given | Change it | Form we are used to |
|---|---|---|
| $y=\frac{3}{2-x}$ | Multiply by -1 top and bottom $y=\frac{3}{2-x}\times\frac{-1}{-1}=\frac{-3}{x-2}$ | $y=\frac{-3}{x-2}$ |

Other types of hyperbola

$$y=\frac{1}{x^2}$$

This graph is called a TRUNCUS

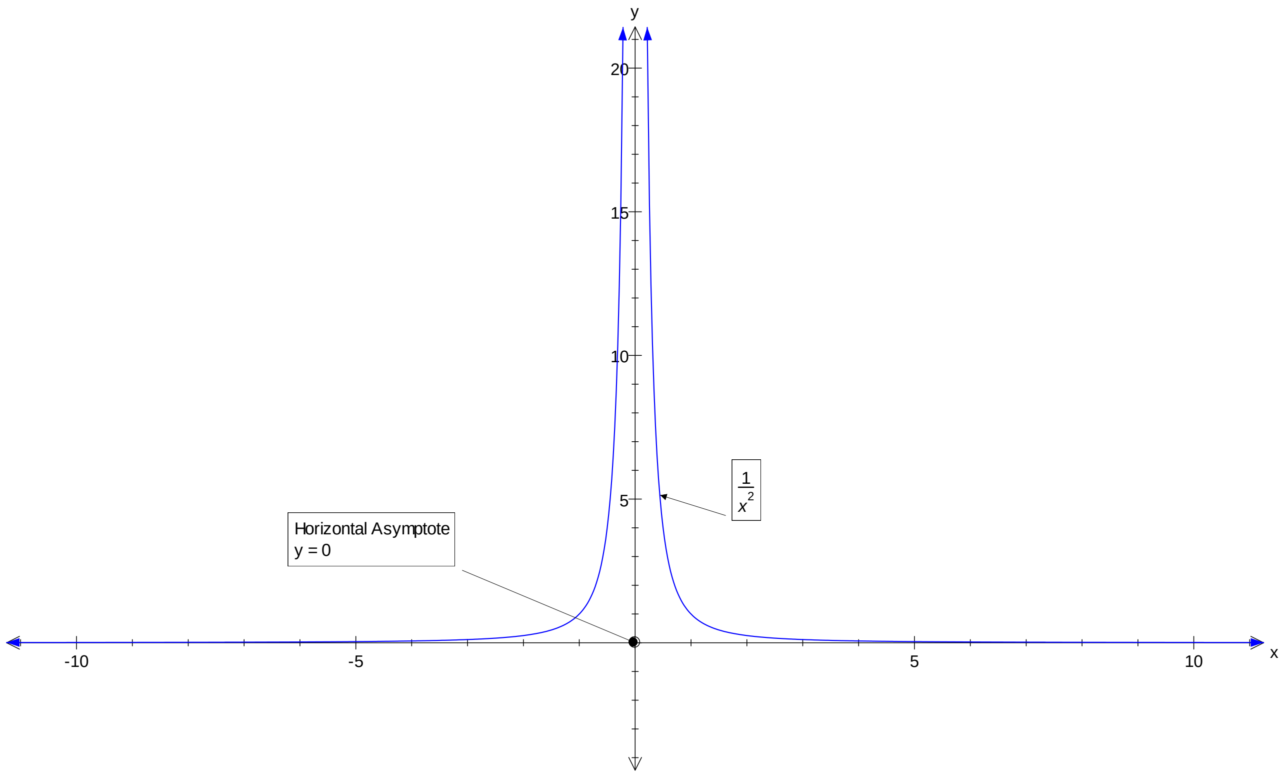
It has a vertical and horizontal asymptote

Vertical asymptote x = 0
Horizontal asymptote y = 0

Domain: R\ {0}

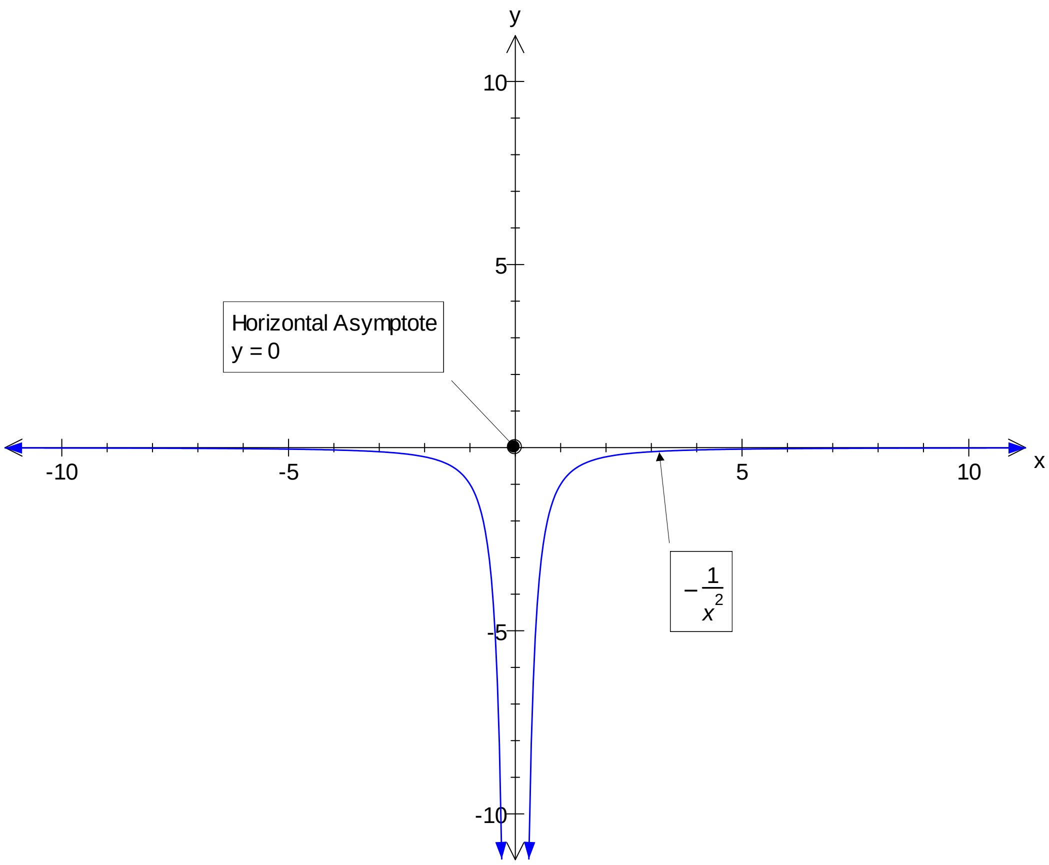
Range: $R^+$

Let's us have a look at the graph of $y=\frac{1}{x^2}$




Now the general case of this graph looks like this $y=\frac{a}{x^2}$

If a > 0 then the graph looks the above. If the a < 0 the graph looks like this

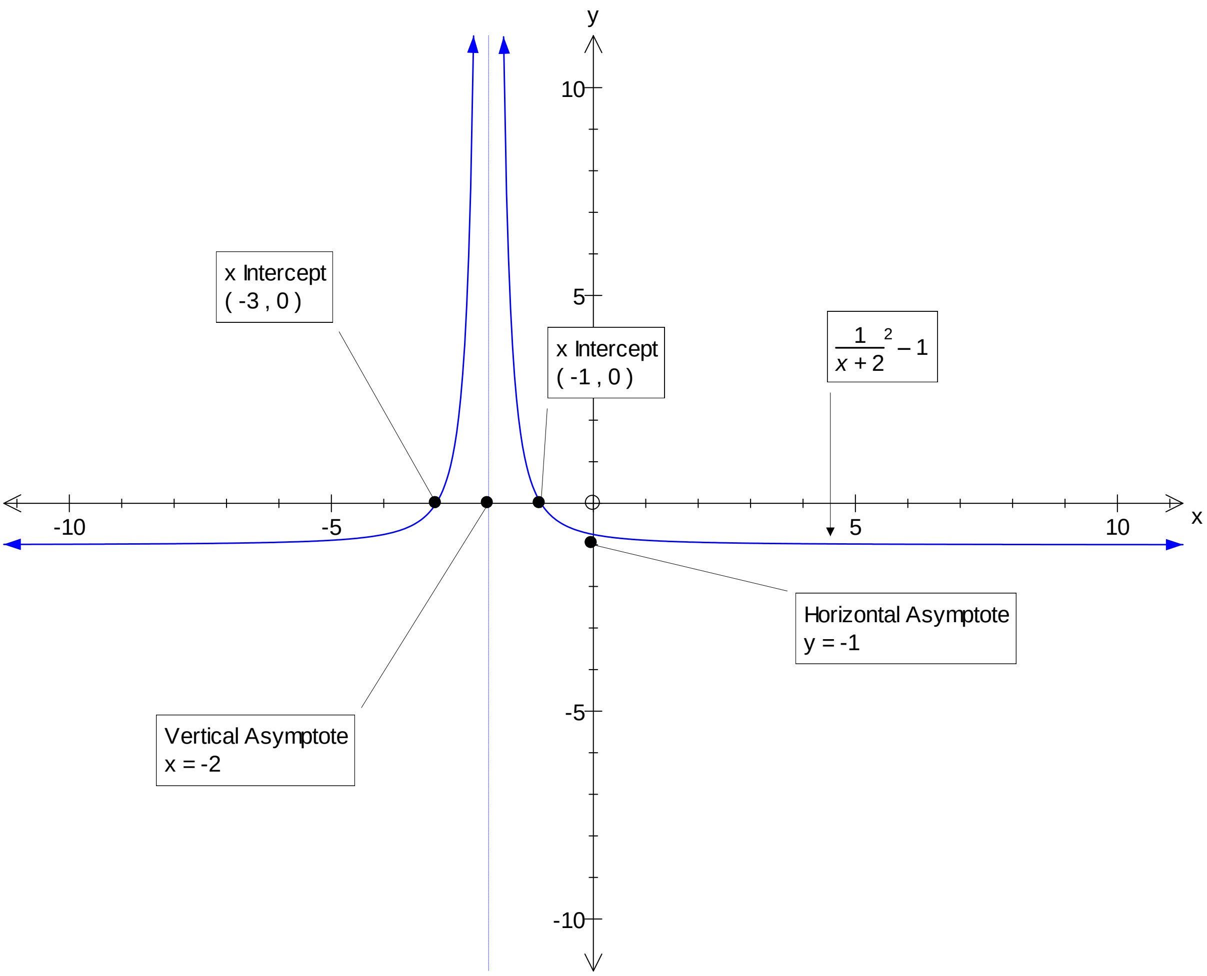



Once again we can have the general case for this type of graph

$$y = \frac{a}{(x-h)^2} + k$$

Now this graph has horizontal asymptote at y=k and a vertical asymptote at x= h

Let's us see another graph $y = \frac{1}{(x+2)^2} - 1$




Notice

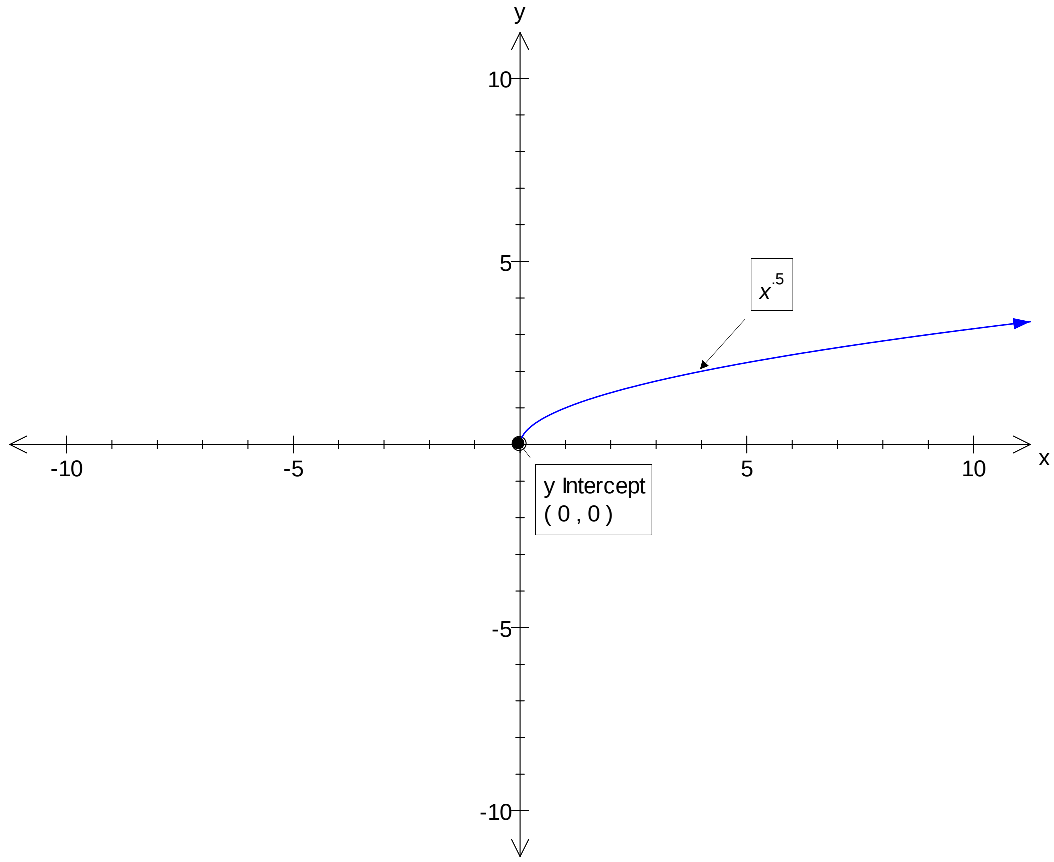
Vertical asymptote at x= -2 Horizontal asymptote at y= -1

Domain: R \ {-2}

Range: y > -1

Another graph to consider is $y = x^{\frac{1}{2}} = \sqrt{x}$

Which is the inverse graph of $y = x^2$




Domain: R U {0}
Range: $R^+$ U {0}

**Power Functions**

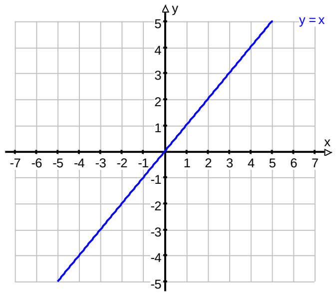
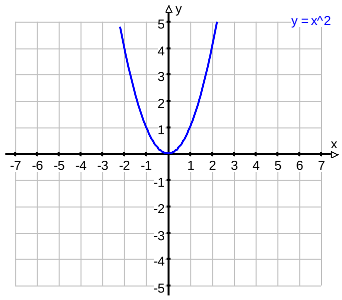
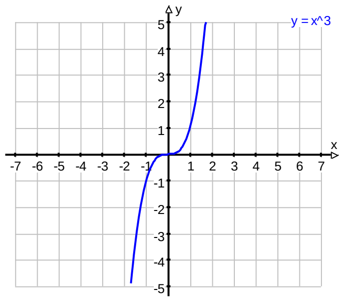
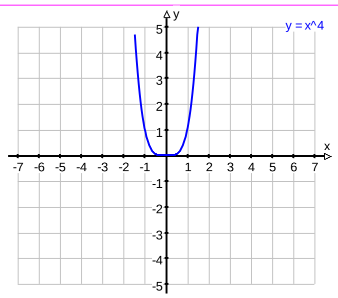
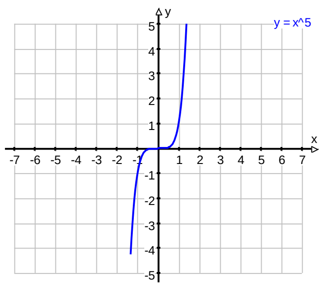
Graph of functions to the power of x are called power functions
Any function with the rule of the form $y = x^n$ , where n is a constant is called a power function.

Example: $y = x$ , $y = x^2$ , $y = x^3$ , $y = x^4$ , $y = x^5$

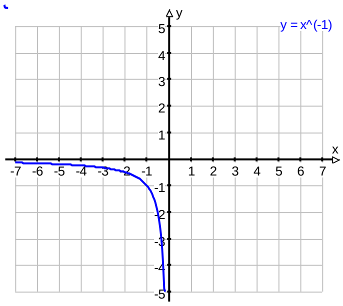
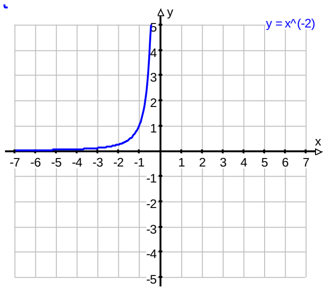
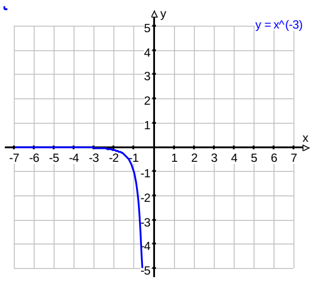
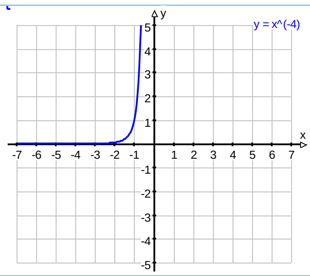
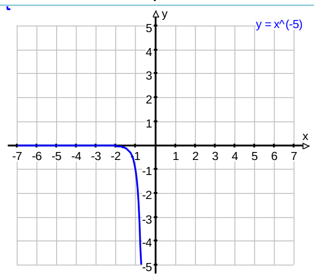
Quick Summary of Properties of Power functions

| | |
|---|---|
| Odd values for the power- $n = 1, 3, 5,$ | The graph produced will have a similar shape to that of a cubic function like $y = x^3$<br>There is a point of inflection at the origin (0,0)<br>As $x \to \infty$ then $y \to \infty$<br>As $x \to -\infty$ then $y \to -\infty$ |
| Even values for the powers | The graphs produced will be similar to $y = x^4$<br>There is a turning point at the origin (0,0)<br>As $x \to \infty$ then $y \to \infty$<br>As $x \to -\infty$ then $y \to -\infty$<br><br>Graphs will pass through (-1,1) and (1,1) and (0,0) |
| Graphs with square roots for example $y = \sqrt{x}$ | This graph is called the square root function. Its domain is valid for $x \geq 0$, thus giving us a range of $y \geq 0$<br><br>It is a parabola on its side essentially. |
| Graphs like this are also square root graphs<br><br>$y = \sqrt{(x+b)} + c$ | Square root graph like above with its endpoint at (-b, c) |
| Graphs like this are also square root graphs<br><br>$y = -\sqrt{(x+b)} + c$ | This is an inverted square root graph with its endpoint at (-b,c) |

Some quick sketches of Positive Power Functions

| Power function $y = x^n$ | Equation of graph | Quick sketch |
|---|---|---|
| $n = 1$ | $y = x^1$<br>linear function |  |
| $n = 2$ | $y = x^2$<br>parabola |  |
| $n = 3$ | $y = x^3$<br>cubic |  |
| $n = 4$ | $y = x^4$<br>quartic |  |
| $n = 5$ | $y = x^5$<br>quintic |  |

Negative Power functions

| Power function $y = x^n$ | Equation of graph | Quick sketch |
| --- | --- | --- |
| $n = -1$ | $y = x^{-1}$ Hyperbola |   |
| $n = -2$ | $y = x^{-2}$ Trucus |   |
| $n = -3$ | $y = x^{-3}$ |   |
| $n = -4$ | $y = x^{-4}$ |   |
| $n = -5$ | $y = x^{-5}$ |   |

**POLYNOMIAL FUNCTIONS**

A polynomial can be represented in the general form by the following expression:

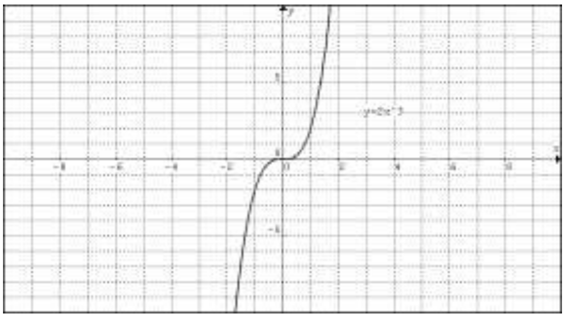
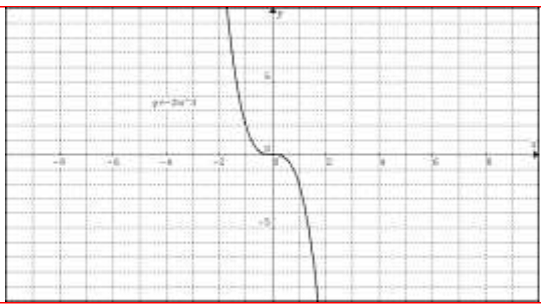
$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots + a_n x^n$$

Where $a_0$ , $a_1$ , $a_2$ , are constants

$a_n x^n$ Is called the leading term and *n* is called the degree of the polynomial.

| Polynomials of different degrees | | |
|---|---|---|
| 1 | $P(x) = 3x^1 + 1$<br>This is called a polynomial of degree 1 | linear functions |
| 2 | $P(x) = 2x^2 + 3x^1 + 1$<br>This is called a polynomial of degree 2 | Quadratic functions |
| 3 | $P(x) = 4x^3 + 5x^2 + 3x^1 + 1$<br>This is called a polynomial of degree 3 | Cubic functions |

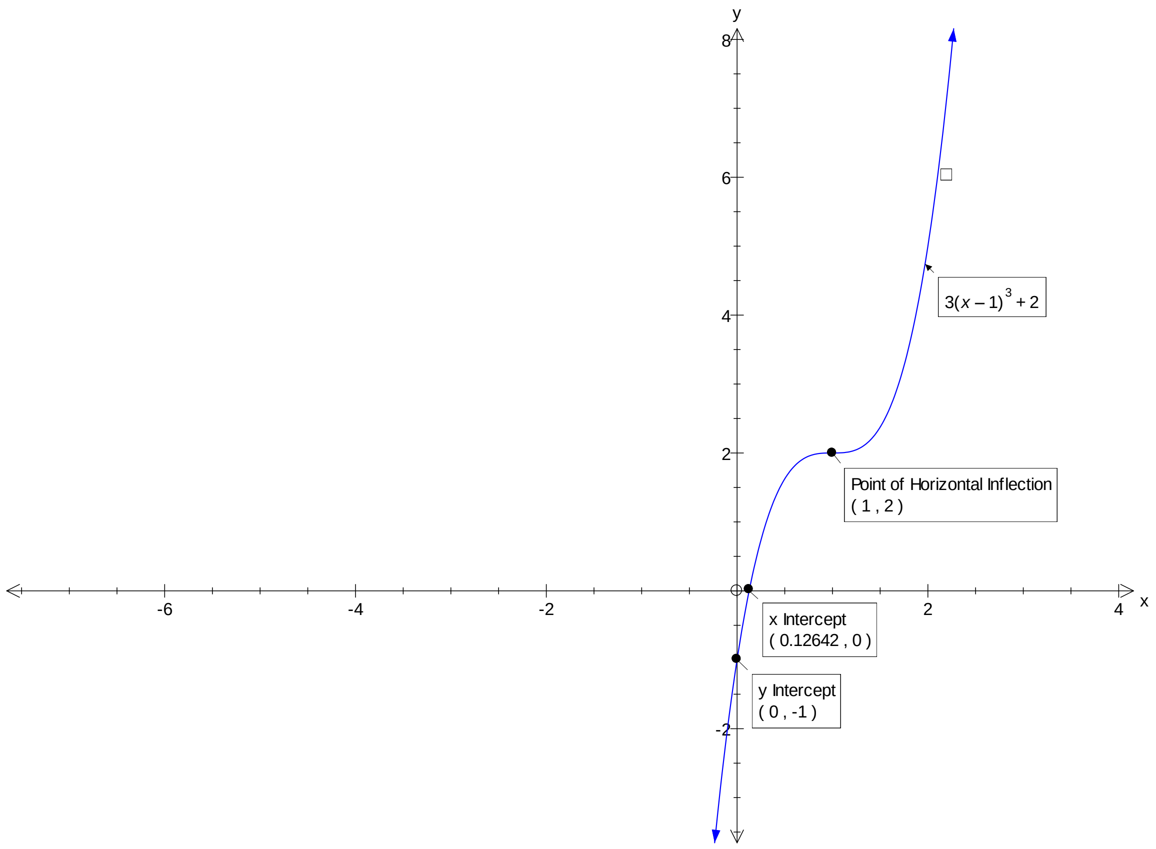
**Review of Polynomials of degree 3**

| Polynomials of degree 3 | |
|---|---|
| General form | $y = ax^3$ |
| Properties | Point of inflexion at (0, 0) |
| if a>0 |  |
| If a < 0 then the graph looks like this |  |

| $y = \pm a(x-h)^3 + k$ | | | | | |
|---|---|---|---|---|---|
| | $\pm$ | $a$ | $h$ | $k$ | Point of inflexion |
| | minus inverts the graph | a-dilation, graph steeper if a > 1 Flatter if 0 < a <1 | horizontal translation of h to right | vertical translation of k | $(h,k)$ |

***Example 3***

Sketch the graph of $y = 3(x-1)^3 + 2$




Also did you notice that compared with the general equation, here a >0 therefore graph is in quadrant 1 and 3.

| Polynomials of the form $y = (x-a)(x-b)(x-c)$ |
|---|
| X intercepts: (a, 0) (b, 0) and (c, 0) |
| Y intercepts: (0, -abc) |
| To find the turning points we will need to differentiate the function and set the derivative to zero and then solve for x |

| Polynomials of the form $y = (x-a)^2(x-b)$ |
| --- |
| X intercepts: (a, 0) (b, 0) |
| Y intercepts: $(0, -a^2b)$ |
| One of the turning points will be at (a, 0) because (x-a) is a repeated factor. |

| Polynomials of the form $y = ax^3 + bx^2 + cx + d$ |
| --- |
| To sketch this type of polynomial we must use the factor theorem and factorize the polynomial to find the x intercepts.<br>If the expression does not factorize, the x intercepts might then be found using a graphics calculator. |
| Y intercept: (0, d) |
| To find the turning points we need to differentiate the function and equate the derivative to zero and then solve for x. |

Graphs of the various polynomials

***Example***

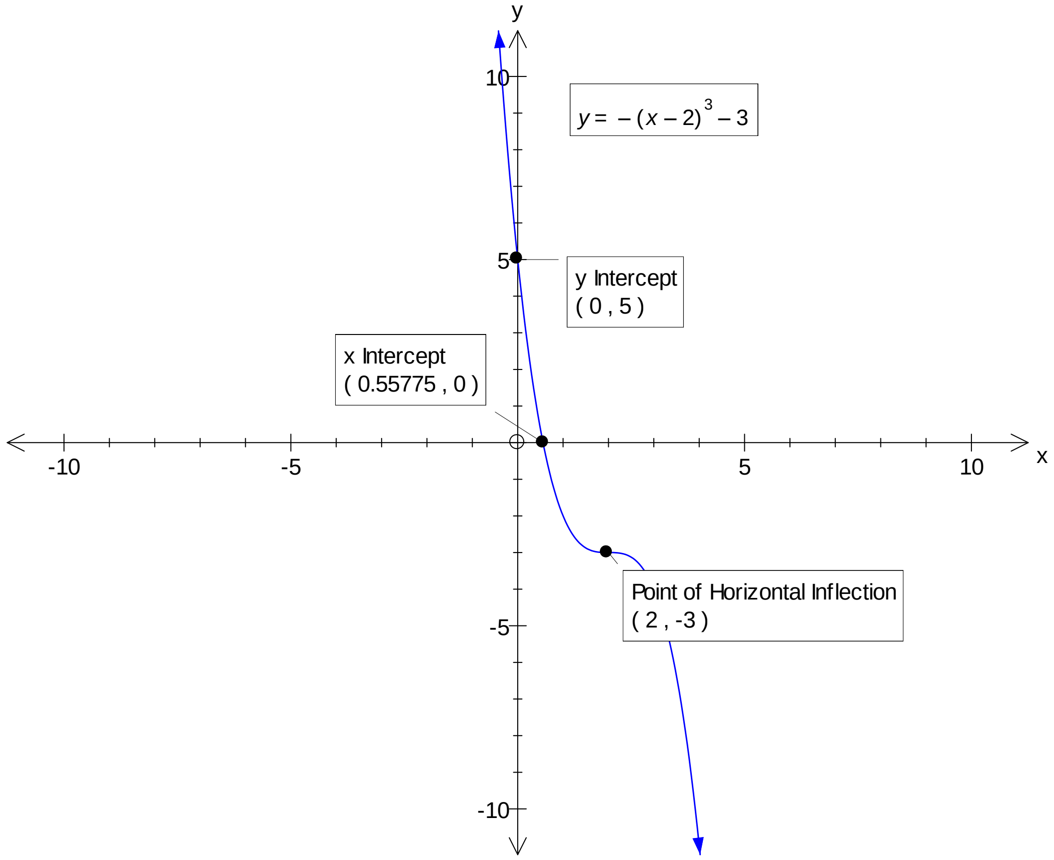
$y = -(x-2)^3 - 3$

Compare this with $y = \pm a(x-h)^3 + k$

Notice $\Rightarrow$ negative sign (-) meaning graph is inverted

$\Rightarrow$ h = 2 and k = -3 therefore (h, k) is (2, -3) is the point of inflection

$\Rightarrow$ so now you could draw the graph roughly




Let us work out the X intercept $\Rightarrow$ let y=0

$$-(x-2)^3 - 3 = 0$$
$$-(x-2)^3 = 3$$
$$(x-2)^3 = -3$$
$$x - 2 = \sqrt[3]{-3}$$
$$x - 2 = -1.4422$$
$$x = 0.5577$$

***Example***

Sketch $y = x^3 - 2x^2 - 5x + 6$

Obviously we could just use our graphics calculator but we would like to use algebra to solve and sketch this graph.

Use the factor theorem to find the X intercepts

$P(x) = x^3 - 2x^2 - 5x + 6$

$P(1) = (1)^3 - 2(1)^2 - 5(1) + 6$

$P(1) = 1 - 2 - 5 + 6$

$P(1) = 0$

Therefore (x -1) is a factor

Now we do long division

$$\begin{array}{r} x^2 - x - 6 \\ (x-1)\overline{\big)\, x^3 - 2x - 5x + 6} \end{array} \quad \to x^3 - 2x^2 - 5x + 6 \to (X-1)(X^2 - X - 6) \to (X-1)(X-3)(X+2)$$

Therefore the x intercepts are x= 1, x=3 and x= -2

Y intercepts (0, 5)

To find the turning points we take the derivative

$\frac{dy}{dx} = 3x^2 - 4x - 5$

$\Rightarrow 3x^2 - 4x - 5 = 0$

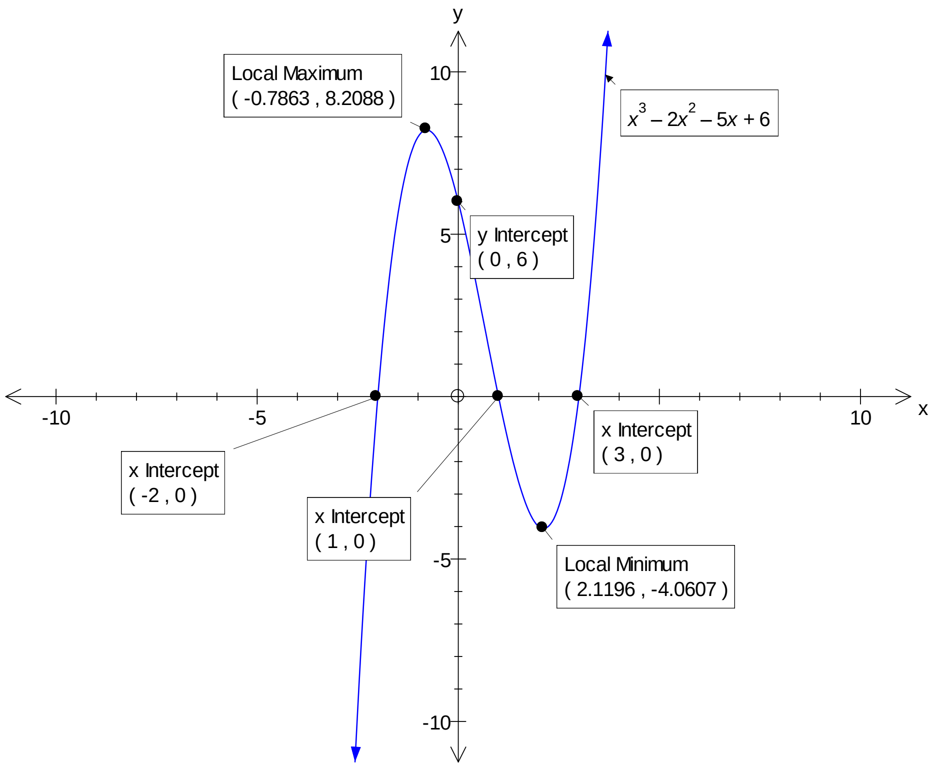
Solve the quadratic using the formula

$x = 2.1196$ or $x = -0.786$

Turning points are therefore

$(2.1196, -4.0606)$ and $(-.79, 8.21)$

Now sketching the graph



Obviously it is faster to work it out with a graphics calculator but there will be questions where you will be required to work things out from first principles

Factorizing Polynomials-Review

**The factor theorem and the remainder theorem applications**

***The factor theorem***

Consider a polynomial $P(x)$ that has $x-\alpha$ as a linear factor. Then $P(x)=(x-\alpha)Q(x)$, where $Q(x)$ is a polynomial one degree lower than $P(x)$, obtained by expanding and comparing coefficients, or dividing $P(x)$ by $x-\alpha$.
Replacing $x$ by $\alpha$ in $P(x)=(x-\alpha)Q(x)$,
$P(\alpha)=(\alpha-\alpha)Q(\alpha)$. Hence $P(\alpha)=0$.

Conversely, for any polynomial $P(x)$, if $P(\alpha)=0$, then $x-\alpha$ is a factor of $P(x)$. This statement is known as the factor theorem, and can be used to find the linear factors of a polynomial if other methods failed.

The factor theorem is best used for polynomials with linear factors of rational coefficients. The value(s) of $\alpha$ is found by trial and error. The possible values of $\alpha$ for trying depend on the first and last coefficients of P(x)

$P(x)=a_0x^n+a_1x^{n-1}+.....+a_{n-1}x+a_n$, $\alpha=\pm\dfrac{a.factor.of.a_n}{a.factor.of.a_0}$.

If all these $\alpha$ values give $P(\alpha)\neq 0$, it does not necessarily mean that $P(x)$ has no linear factors, because the coefficients of the linear factor(s) may be irrational.

**Example 1:** Use the factor theorem to find a linear factor of $3x^3-2x^2-7x-2$, then find the quadratic factor and hence all the linear factors.

Let $P(x)=3x^3-2x^2-7x-2$. The possible values of $\alpha$ for testing are $\alpha=\pm\dfrac{1,2}{1,3}$, i.e. $\alpha=\pm\dfrac{1}{3},\pm\dfrac{2}{3},\pm1,\pm2$.

$P(1)=3(1)^3-2(1)^2-7(1)-2\neq 0$.
$P(-1)=3(-1)^3-2(-1)^2-7(-1)-2=0$, $\therefore x+1$ is a factor.
Divide $P(x)$ by $x+1$ to find the quadratic factor.

$$\begin{array}{r}
3x^2-5x-2 \\
x+1)\overline{\,3x^3-2x^2-7x-2} \\
-(3x^3+3x^2) \\
\hline
-5x^2-7x \\
-(-5x^2-5x) \\
\hline
-2x-2 \\
-(-2x-2) \\
\hline
0
\end{array}$$

Hence $P(x)=(x+1)(3x^2-5x-2)=(x+1)(3x+1)(x-2)$.

Another possible outcome:

$P\left(-\frac{1}{3}\right)=3\left(-\frac{1}{3}\right)^3-2\left(-\frac{1}{3}\right)^2-7\left(-\frac{1}{3}\right)-2=0$, $\therefore x+\frac{1}{3}$ is a factor.

$$
\begin{array}{rl}
 & 3x^2-3x-6 \\
\cline{2-2}
x+\frac{1}{3}) & 3x^3-2x^2-7x-2 \\
 & -\left(3x^3+x^2\right) \\
\cline{2-2}
 & -3x^2-7x \\
 & -\left(-3x^2-x\right) \\
\cline{2-2}
 & -6x-2 \\
 & -\left(-6x-2\right) \\
\cline{2-2}
 & 0
\end{array}
$$

Hence $P(x)=\left(x+\frac{1}{3}\right)\left(3x^2-3x-6\right)=3\left(x+\frac{1}{3}\right)\left(x^2-x-2\right)$

$=3\left(x+\frac{1}{3}\right)(x-2)(x+1)$. It is equivalent to the previous result.

**Example 2:** Given $x+1$ is a factor of $3x^4+x^3-9x^2-9x-2$, find the cubic factor $Q(x)$ such that

$$3x^4+x^3-9x^2-9x-2=(x+1)Q(x).$$

Let $Q(x)=ax^3+bx^2+cx+d$, then

$3x^4+x^3-9x^2-9x-2=(x+1)\left(ax^3+bx^2+cx+d\right)$.

Expand to obtain $3x^4+x^3-9x^2-9x-2$

$=ax^4+(a+b)x^3+(b+c)x^2+(c+d)x+d$.

Compare the coefficients on both sides, $a=3$, $a+b=1$, $b+c=-9$, $c+d=-9$ and $d=-2$. Hence $b=-2$, $c=-7$.

$\therefore Q(x)=3x^3-2x^2-7x-2$.

This example illustrates an alternative method in finding $Q(x)$ to long division shown in the last example.

**Example 3:** Use the factor theorem to find the linear factors of $x^4-x^3+x^2-3x+2$.

Let $P(x)=x^4-x^3+x^2-3x+2$. Test $\alpha=\pm1,\pm2$.

$P(-1)=(-1)^4-(-1)^3+(-1)^2-3(-1)+2\neq 0$

$P(1)=(1)^4-(1)^3+(1)^2-3(1)+2=0$, $\therefore x-1$ is a factor.

Hence $P(x)=(x-1)Q(x)$.

Use long division or comparing coefficients to find $Q(x)=x^3+x-2$.

Hence $P(x)=(x-1)\left(x^3+x-2\right)$. Use the factor theorem on $Q(x)=x^3+x-2$ to find its linear factor. Test $\alpha=\pm1,\pm2$.

$Q(1)=(1)^3+(1)-2=0$, $\therefore x-1$ is a factor of $Q(x)$.

Hence $P(x)=(x-1)(x-1)T(x)$. $T(x)$ is quadratic and found by long division of $P(x)$ by the expansion of $(x-1)(x-1)$, or comparison of coefficients as discussed in example 2, $T(x)=x^2+x+2$.

$\therefore P(x)=(x-1)(x-1)\left(x^2+x+2\right)$.

**The remainder theorem** When a polynomial $P(x)$ is divided by a linear binomial $\alpha x - \beta$, the remainder can be found quickly without actually carrying out the division.

Since

| | | |
|---|---|---|
| | $Q(x)$ | **Quotient** |
| $ax - \beta$ | $P(x)$ | **Dividend (polynomial)** |
| **Divisor** | | |
| | $R$ | Remainder |

$\therefore P(x) = (\alpha x - \beta)Q(x) + R$.

When $x = \dfrac{\beta}{\alpha}$, $P\left(\dfrac{\beta}{\alpha}\right) = \left(\alpha\left(\dfrac{\beta}{\alpha}\right) - \beta\right)Q(x) + R = R$,

i.e. the remainder $R = P\left(\dfrac{\beta}{\alpha}\right)$ when $P(x)$ is divided by $\alpha x - \beta$. This is known as the remainder theorem.

**Example 4:** Find the remainder when $P(x) = 2x^4 - x^3 + 5x - 11$ is divided by (i) $x + 5$, (ii) $2x - 3$, (iii) $x - 2a$.

(i) $R = P(-5) = 2(-5)^4 - (-5)^3 + 5(-5) - 11 = 1339$

(ii) $R = P\left(\dfrac{3}{2}\right) = 2\left(\dfrac{3}{2}\right)^4 - \left(\dfrac{3}{2}\right)^3 + 5\left(\dfrac{3}{2}\right) - 11 = \dfrac{13}{4}$

(iii) $R = P(2a) = 2(2a)^4 - (2a)^3 + 5(2a) - 11 = 32a^4 - 8a^3 + 10a - 11$

**Example 5:** Given that $x - 2$ is a factor of $3x^3 + px^2 + qx - 2$ and the remainder is $-20$ when the cubic polynomial is divided by $x + 2$. Find the values of *p* and *q*.

Use the factor theorem and the remainder theorem to set up two simultaneous equations for *p* and *q*.
Let $P(x) = 3x^3 + px^2 + qx - 2$.
$x - 2$ is a factor of $P(x)$, $\therefore P(2) = 0$,
$\therefore 3(2)^3 + p(2)^2 + q(2) - 2 = 0$, $\qquad \therefore 2p + q = -11$ ……(1)
The remainder is $-20$ when divided by $x + 2$, $\therefore P(-2) = -20$,
$\therefore 3(-2)^3 + p(-2)^2 + q(-2) - 2 = -20$, $\qquad \therefore 2p - q = 3$ ……(2)

Solve eqs (1) and (2) for *p* and *q*.
(1) + (2), $4p = -8$, $\therefore p = -2$
(1) – (2), $2q = -14$, $\therefore q = -7$

**Special Notes**

$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

$x^2 - y^2 = (x - y)(x + y)$